

REDUCTIVE GROUPS, EPSILON FACTORS AND WEIL INDICES

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1. INTRODUCTION

1.1. Notation pertaining to the ground field F . We work over a locally compact field F , assumed to be nondiscrete, and we fix a nontrivial character $\psi : F \rightarrow \mathbb{C}^\times$ on the additive group F . We make no restrictions on the characteristic of F . Finally we choose a separable algebraic closure \overline{F} of F and put $\Gamma := \text{Gal}(\overline{F}/F)$.

1.2. Notation pertaining to the group G . We consider a connected reductive group G over F . We write G_{ad} for its adjoint group and G_{sc} for the simply connected cover of the derived group of G . There are then canonical homomorphisms

$$G_{\text{sc}} \rightarrow G \rightarrow G_{\text{ad}}.$$

We choose a quasi-split inner form G_0 of G . We also choose a maximal F -torus T_0 in G_0 having the property that there exists a Borel F -subgroup B_0 containing it. It is well-known that T_0 is unique up to conjugacy under $G_0(F)$.

1.3. Three ingredients in the main theorem.

1.3.1. *First ingredient.* The first ingredient is the sign $e(G) \in \{\pm 1\}$ attached to G in [Kot83].

Now let T be a maximal F -torus in G . The remaining two ingredients are complex fourth roots of unity that depend on (G, T) .

1.3.2. *Second ingredient.* The Galois group Γ acts on the cocharacter group $X_*(T)$. Of course the action is trivial on some open subgroup of Γ . In any event $X_*(T)_{\mathbb{R}}$ is a real representation of Γ , and $X_*(T)_{\mathbb{C}}$ is a complex orthogonal representation of Γ .

We are interested in the complex orthogonal representations $X_*(T)_{\mathbb{C}}$ and $X_*(T_0)_{\mathbb{C}}$. Their difference $X_*(T)_{\mathbb{C}} - X_*(T_0)_{\mathbb{C}}$ is a virtual orthogonal

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representation whose dimension is 0. The second ingredient is the local epsilon factor

$$\epsilon_L(X_*(T)_{\mathbb{C}} - X_*(T_0)_{\mathbb{C}}, \psi).$$

Here ϵ_L is as in Tate's article [Tat79]: the subscript L indicates that epsilon factors are formed using the conventions of Langlands rather than those of Deligne. It would make no difference if we used $X^*(T)$ rather than $X_*(T)$, since we are dealing with self-contragredient representations.

Deligne's results [Del76] on local epsilon factors of virtual orthogonal complex representations of Γ are essential to our proof of the main theorem, as is a result of Jacquet-Langlands [JL70] giving an equality between certain local epsilon factors and certain Weil indices.

1.3.3. Third ingredient. The third ingredient is the Weil index $\gamma(Q_V, \psi)$ (see [Wei64]) of a certain even dimensional, nondegenerate quadratic space (V, Q_V) over F . Because the dimension is even, the Weil index is a complex fourth root of unity.

The quadratic form Q_V depends on (G, T) , and it lives on the F -vector space V defined as the unique T -invariant complement to $\mathfrak{t} := \text{Lie}(T)$ in $\mathfrak{g} := \text{Lie}(G)$. Equivalently, V is the unique T_{sc} -invariant complement to \mathfrak{t}_{sc} in \mathfrak{g}_{sc} . (As is customary, T_{sc} denotes the preimage of T in G_{sc} .)

How is the quadratic form Q_V defined? Giving an explicit formula for it is possible only over the separable closure \overline{F} . So, just for a moment, we extend scalars to \overline{F} . We then have the root space decomposition

$$V = \bigoplus_{\alpha} \mathfrak{g}_{\alpha}.$$

Even more useful to us is the coarser decomposition

$$V = \bigoplus_{\pm\alpha} \mathfrak{g}_{\pm\alpha},$$

where $\mathfrak{g}_{\pm\alpha} := \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha}$. Our quadratic form Q_V will be defined as the direct sum of quadratic forms $Q_{\pm\alpha}$ on the planes $\mathfrak{g}_{\pm\alpha}$ over \overline{F} .

The value of $Q_{\pm\alpha}$ on $v \in \mathfrak{g}_{\pm\alpha}$ is defined as follows. We decompose v as the sum of $v_+ \in \mathfrak{g}_{\alpha}$ and $v_- \in \mathfrak{g}_{-\alpha}$. We regard v_+ and v_- as elements in \mathfrak{g}_{sc} , and then form their Lie bracket $[v_+, v_-] \in \mathfrak{t}_{\text{sc}}$. It is a standard fact that $[v_+, v_-]$ is a scalar multiple of the coroot for α , and we take this scalar (in \overline{F}) as the definition of $Q_{\pm\alpha}(v)$. (Even when the characteristic is 2, the coroot H_{α} is nonzero in \mathfrak{t}_{sc} , which is why we moved from \mathfrak{g} to \mathfrak{g}_{sc} before taking the Lie bracket.)

In summary, we define $Q_{\pm\alpha}(v)$ to be the unique scalar $c \in \overline{F}$ such that

$$[v_+, v_-]_{\text{sc}} = cH_\alpha,$$

where the subscript sc indicates that the Lie bracket is taken in $\mathfrak{g}_{\text{sc}} = \mathfrak{t}_{\text{sc}} \oplus V$, and $H_\alpha \in \mathfrak{t}_{\text{sc}}$ is the coroot for α . If we replace α by $-\alpha$, both sides of this equality are replaced by their negatives, but c does not change. So $Q_{\pm\alpha}$ depends only on the unordered pair $\{\alpha, -\alpha\}$. It is a canonically defined nondegenerate quadratic form on the plane $\mathfrak{g}_{\pm\alpha}$.

Now $Q_{\pm\alpha}$ is defined only over \overline{F} , but the direct sum

$$Q_V := \bigoplus_{\pm\alpha} Q_{\pm\alpha}$$

is defined over F , and our discussion of the third ingredient is complete.

1.4. Statement of the main theorem. For any (G, T) over F , we want to prove the following statement, an application of which can be found in Kaletha's paper [Kal15].

Theorem 1.1. *There is an equality*

$$e(G)\gamma(Q_V, \psi) = \epsilon_L(X_*(T)_{\mathbb{C}} - X_*(T_0)_{\mathbb{C}}, \psi).$$

1.5. Organization of the paper. In what follows we first sketch a proof of the main theorem, and then fill in the details. The body of the paper is not entirely self-contained, and the reader will need to be familiar with the material in the appendices, where some of the key notation is introduced.

The appendices are largely review, so many proofs are omitted. However Proposition F.1, which gives a convenient method to recognize when Weil indices and local epsilon factors of orthogonal representations are equal, may be new. The same is true of Proposition G.2 for fields of characteristic 2. For fields of characteristic different from 2, Proposition G.2 is a result of Fröhlich [Frö85].

2. SKETCH OF THE PROOF OF THE MAIN THEOREM

2.1. Reduction steps. We refer to the statement in the theorem as $P(G, T)$ in the reduction steps that follow.

2.1.1. Reduction 1. The statement $P(G, T)$ is true if and only if the statement $P(G_{\text{ad}}, T_{\text{ad}})$ is true. Indeed, each of three quantities appearing in the statement remains unchanged when we pass from (G, T) to $(G_{\text{ad}}, T_{\text{ad}})$. (As is customary T_{ad} is the maximal torus in G_{ad} whose preimage in G is T .)

2.1.2. *Reduction 2.* If $P(G_i, T_i)$ is true for $i = 1, 2$, then $P(G_1 \times G_2, T_1 \times T_2)$ is true. Indeed, each of the three quantities appearing in the statement is multiplicative in the pair (G, T) .

2.1.3. *Reduction 3.* Let E/F be a finite separable extension, and use $\psi \circ \text{tr}_{E/F}$ as additive character on E . Then, for any pair (G, T) over E , statement $P(G, T)$ is true if and only if statement $P(R_{E/F}G, R_{E/F}T)$ is true. Here $R_{E/F}$ denotes Weil restriction of scalars. Indeed, each of the three relevant quantities is preserved by Weil restriction of scalars. (We are using that local epsilon factors are inductive for virtual representations of degree 0.)

2.1.4. *Reduction 4.* It is well-known that T transfers to a maximal torus T' in the quasi-split inner form G_0 . Then, for any choice of T' (which is only well-defined up to stable conjugacy), the statement $P(G, T)$ is true if and only if the statement $P(G_0, T')$ is true.

This reduction is a bit more complicated than the others. The local epsilon factor is of course the same for (G, T) and (G_0, T') , and moreover $e(G_0) = 1$. So we need to show that

$$e(G) = \gamma(Q_{V'}, \psi) \gamma(Q_V, \psi)^{-1}.$$

This statement is similar to a result of Gan [Gan98] relating $e(G)$ to the Killing form.

Now the difference between T and T' is measured by an invariant $t \in H^1(F, T)$. By Corollary H.2 we just need to show that $e(G) \in \text{Br}_2(F) = \{\pm 1\}$ is equal to the image of t under the map

$$\partial_U : H^1(F, T) \rightarrow \text{Br}_2(F)$$

arising from the fppf short exact sequence

$$1 \rightarrow \mu_2 \rightarrow U \rightarrow T \rightarrow 1$$

corresponding to the element $\lambda_V \in X^*(T)/2X^*(T)$ defined in the discussion leading up to Corollary H.2. It is evident that λ_V works out to the sum of the positive roots (for any choice of positive system), and therefore gives rise to the same connecting map as the one used to define $e(G)$.

2.2. Using the reduction steps to get down to the essential case. How do we use these reduction steps? Using the first one, we may assume that G is adjoint. Then G is a product of F -simple groups. By the second reduction we may now assume that G is F -simple. Then (G, T) is of the form $(R_{E/F}H, R_{E/F}T_H)$ for some E/F and some (H, T_H) over E with H absolutely simple. By the third reduction we may

now assume that G is an absolutely simple group. Finally, the fourth reduction allows us to assume that G is also quasi-split over F .

Because G is quasi-split, $e(G) = 1$ and the main theorem reduces to the statement that

$$(2.1) \quad \gamma(Q_V, \psi) = \epsilon_L(X_*(T)_{\mathbb{C}} - X_*(T_0)_{\mathbb{C}}, \psi).$$

To prove this it suffices to check that the number

$$(2.2) \quad \gamma(Q_V, \psi) \epsilon_L(X_*(T)_{\mathbb{C}}, \psi)^{-1}$$

is independent of the maximal torus T in G . Indeed, when $T = T_0$ the number (2.2) reduces to $\epsilon_L(X_*(T)_{\mathbb{C}}, \psi)^{-1}$, because the Weil index of V_0 is automatically 1. Here V_0 is the analog of V for T_0 . (We have $V_0 = V_+ \oplus V_-$, where V_+ comes from B_0 -positive roots and V_- comes from B_0 -negative roots. Both V_+ and V_- are defined over F , and the quadratic form Q_{V_0} vanishes identically on both of them. So V_0 is a direct sum of hyperbolic planes, and its Weil index is 1.)

2.3. Rough sketch. We finish this section by giving a rough sketch of the rest of the proof. In subsequent sections we will fill in all the details. We are now working with a quasi-split absolutely simple adjoint group G . For any two maximal F -tori T, \tilde{T} in G we must show that

$$\gamma(Q_{\tilde{V}}, \psi) \gamma(Q_V, \psi)^{-1} = \epsilon_L(X_*(\tilde{T})_{\mathbb{C}} - X_*(T)_{\mathbb{C}}, \psi).$$

Here, of course, $\mathfrak{g} = \tilde{\mathfrak{t}} \oplus \tilde{V}$ is the analog for \tilde{T} of the decomposition $\mathfrak{g} = \mathfrak{t} \oplus V$.

How will we do this? The details depend on the type of the root system R of G . Types A, D, E (the simply laced ones) are the easiest to handle. Types B, C, F_4 will require extra effort in characteristic 2, and G_2 will require extra effort in characteristic 3, but in other characteristics the non-simply laced cases can be treated by a slight elaboration of the method used to handle types A, D, E .

Curiously enough the group E_8 is easiest of all: besides being simply laced, it has the property that the adjoint group is also simply connected. So we begin our sketch by considering the case of E_8 . Then there is a canonical G -invariant nondegenerate quadratic form Q_G on \mathfrak{g} having the property that, for every maximal torus T in G , the restriction of Q_G to V is equal to Q_V . We then have

$$\gamma(Q_G|_{\mathfrak{t}}, \psi) \gamma(Q_V, \psi) = \gamma(Q_G, \psi) = \gamma(Q_G|_{\tilde{\mathfrak{t}}}, \psi) \gamma(Q_{\tilde{V}}, \psi),$$

from which we conclude that

$$\gamma(Q_{\tilde{V}}, \psi) \gamma(Q_V, \psi)^{-1} = \gamma(Q_G|_{\mathfrak{t}}, \psi) \gamma(Q_G|_{\tilde{\mathfrak{t}}}, \psi)^{-1}.$$

To complete the argument it remains only to show that

$$\gamma(Q_G|_{\mathfrak{t}}, \psi)\gamma(Q_G|_{\bar{\mathfrak{t}}}, \psi)^{-1} = \epsilon_L(X_*(\tilde{T})_{\mathbb{C}} - X_*(T)_{\mathbb{C}}, \psi).$$

For this (see Corollary F.2) we need to show that

$$HW(Q_G|_{\bar{\mathfrak{t}}}, Q_G|_{\mathfrak{t}}) = \overline{SW}(X_*(\tilde{T})_{\mathbb{C}} - X_*(T)_{\mathbb{C}}).$$

A general result of Fröhlich (see subsection G.4) can be used to show that the Hasse-Witt invariant on the left is equal to the Stiefel-Whitney class on the right, plus a correction term whose vanishing we must check. (In fact we will see that the correction term vanishes in all simply laced cases. In the non-simply laced cases it need not vanish, but it is canceled by another correction term coming from the fact that the restriction of Q_G to V is no longer equal to Q_V .)

For other root systems there is no quadratic form Q_G as well-behaved as the one for E_8 . Even in type A_n , we don't really want to work with the adjoint group PGL_n , and the simply connected form SL_n isn't much better. We could use GL_n , but we prefer a systematic construction that works perfectly for all simply laced groups and makes a significant improvement even in the non-simply laced case.

As a result of our reduction process, we happened to end up with a group of adjoint type. But, as we saw in the first reduction step, for general (G, T) the truth of $P(G, T)$ is the same as that of $P(G_{\text{ad}}, T_{\text{ad}})$. Since every maximal torus in G_{ad} is of the form T_{ad} for a unique maximal torus T of G , we are free to replace the adjoint form by one that is more convenient. Our choice will be one that has been used in various settings, e.g. Vinberg's work [Vin95] on reductive monoids.

So for the remainder of this sketch G will no longer be adjoint. It will still be quasi-split with irreducible root system, and, because Vinberg's construction yields groups with especially favorable properties, \mathfrak{g} will admit a canonical G -invariant quadratic form Q_G that is often nondegenerate. It will turn out that Q_G is degenerate only for B, C, F_4 in characteristic 2 and G_2 in characteristic 3.

In fact, now that we have moved from the adjoint form to a better behaved one, the approach sketched for E_8 continues to work for all groups of type A, D, E . For the other types, in which there are long roots and short roots, it is no longer true that the restriction of Q_G to V coincides with Q_V . In fact, V breaks up as a direct sum of two subspaces, one coming from the short roots, and one from the long roots. On the part coming from the long roots, Q_G and Q_V agree. On the part coming from the short roots, Q_G agrees with ℓQ_V , where ℓ is 2 in types B, C, F_4 and is 3 in type G_2 . As long as the characteristic

of F is different from ℓ , a slight modification of the approach used for A, D, E still works.

When the characteristic of F is equal to ℓ , the quadratic form Q_G is degenerate, and the representation of G on \mathfrak{g} is reducible. In fact there is a canonical short exact sequence

$$0 \rightarrow \mathfrak{g}'' \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}' \rightarrow 0$$

of G -modules, and we can achieve our goal by constructing nondegenerate quadratic forms $Q_{\mathfrak{g}'}, Q_{\mathfrak{g}''}$ on $\mathfrak{g}', \mathfrak{g}''$ respectively, and then working with $(\mathfrak{g}', Q_{\mathfrak{g}'})$ and $(\mathfrak{g}'', Q_{\mathfrak{g}''})$ rather than (\mathfrak{g}, Q_G) .

This completes our sketch. Now we resume giving a detailed proof. We start with some preliminary constructions over \mathbb{Z} .

3. DISCUSSION OF THE PINNED GROUP \mathbb{G} OVER \mathbb{Z}

As we have seen, it is enough to treat quasi-split adjoint groups with irreducible root system. Such a group is obtained by twisting a split adjoint group using a homomorphism from Γ to the group of automorphisms of the Dynkin diagram. But, as we mentioned above, we are not obliged to stick with adjoint groups, and in fact it is advantageous to use Vinberg's construction, which produces groups G that are symmetrically located between G_{sc} and G_{ad} . We also need to construct the relevant quadratic forms, and for this it is best to start with pinned groups over \mathbb{Z} . Only later we will extend scalars to F and make an outer twist.

3.1. Definition of the group \mathbb{G} . We begin with a pinned group

$$(\mathbb{G}_{\text{sc}}, \mathbb{B}_{\text{sc}}, \mathbb{T}_{\text{sc}}, \{\eta_\alpha\})$$

over \mathbb{Z} with \mathbb{G}_{sc} semisimple and simply connected. We use the same system of notation as in Appendix I, so the reader should at least glance at that appendix before continuing to read this section.

We assume that the root system R of \mathbb{T}_{sc} in \mathbb{G}_{sc} is irreducible. To R is associated $\ell \in \{1, 2, 3\}$ (see Appendix I).

Consider the connected reductive group \mathbb{G} over \mathbb{Z} obtained by putting

$$\mathbb{G} := (\mathbb{G}_{\text{sc}} \times \mathbb{T}_{\text{sc}}) / Z(\mathbb{G}_{\text{sc}}),$$

where $Z(\mathbb{G}_{\text{sc}})$ is the center of \mathbb{G}_{sc} , embedded in $(\mathbb{G}_{\text{sc}} \times \mathbb{T}_{\text{sc}})$ via $z \mapsto (z, z)$. We will also need the maximal torus $\mathbb{T} = (\mathbb{T}_{\text{sc}} \times \mathbb{T}_{\text{sc}}) / Z(\mathbb{G}_{\text{sc}})$. The group \mathbb{G} is useful for our purposes because, as we will see, there is an especially well-behaved quadratic form $Q_{\mathbb{G}}$ on the free abelian group $\text{Lie}(\mathbb{G})$.

Observe that

$$(3.1) \quad X_*(\mathbb{T}) = \{(x_1, x_2) \in X_*(\mathbb{T}_{\text{ad}}) \times X_*(\mathbb{T}_{\text{ad}}) : x_1 - x_2 \in X_*(\mathbb{T}_{\text{sc}})\},$$

$$(3.2) \quad X^*(\mathbb{T}) = \{(x_1, x_2) \in X^*(\mathbb{T}_{\text{sc}}) \times X^*(\mathbb{T}_{\text{sc}}) : x_1 + x_2 \in X^*(\mathbb{T}_{\text{ad}})\},$$

where \mathbb{T}_{ad} is the maximal torus $\mathbb{T}_{\text{sc}}/Z(\mathbb{G}_{\text{sc}})$ in the adjoint group \mathbb{G}_{ad} of \mathbb{G} . Here, as usual, we are using $\mathbb{T}_{\text{sc}} \rightarrow \mathbb{T}_{\text{ad}}$ to identify the root lattice $X^*(\mathbb{T}_{\text{ad}})$ with a subgroup of finite index in the weight lattice $X^*(\mathbb{T}_{\text{sc}})$, and similarly for the coroot lattice inside the coweight lattice.

We also put $\mathbb{B} := (\mathbb{B}_{\text{sc}} \times \mathbb{T}_{\text{sc}})/Z(\mathbb{G}_{\text{sc}})$, a Borel subgroup of \mathbb{G} .

3.2. Definition of $Q_{\mathbb{T}}$. As in Appendix I we consider the \mathbb{Q} -vector space

$$\mathfrak{a} := X_*(\mathbb{T}_{\text{sc}})_{\mathbb{Q}} = X_*(\mathbb{T}_{\text{ad}})_{\mathbb{Q}},$$

and the quadratic form Q_1 on it. We define a canonical quadratic form $Q_{\mathbb{T}} : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathbb{Q}$ by putting

$$Q_{\mathbb{T}}(x_1, x_2) := Q_1(x_1) - Q_1(x_2).$$

We write $B_{\mathbb{T}}$ for the symmetric bilinear form obtained by polarizing $Q_{\mathbb{T}}$. So $B_{\mathbb{T}}(x, x) = 2Q_{\mathbb{T}}(x)$ and $Q_{\mathbb{T}}(x + y) = Q_{\mathbb{T}}(x) + Q_{\mathbb{T}}(y) + B_{\mathbb{T}}(x, y)$. We also view $B_{\mathbb{T}}$ as an isomorphism from $\mathfrak{a} \times \mathfrak{a}$ to its linear dual.

The form $Q_{\mathbb{T}}$ is invariant under the natural action of $\Omega \times \Omega$ on $\mathfrak{a} \times \mathfrak{a}$, because Q_1 is invariant under Ω . (See Appendix I for the definition of $\Omega = W \rtimes \Omega_0$.)

3.3. Integrality properties of $Q_{\mathbb{T}}$. We see from equation (3.1) that $\Lambda := X_*(\mathbb{T})$ is a lattice in the vector space $\mathfrak{a} \times \mathfrak{a}$. We need to understand the integrality properties of $Q_{\mathbb{T}}$ relative to Λ . We denote by Λ^{\perp} the lattice perpendicular to Λ . By definition, we have

$$\Lambda^{\perp} = \{x \in \mathfrak{a} \times \mathfrak{a} : B_{\mathbb{T}}(x, x') \in \mathbb{Z} \quad \forall x' \in \Lambda\}.$$

Now $X^*(\mathbb{T})$ is a lattice in $\mathfrak{a}^* \times \mathfrak{a}^*$, and it follows immediately from the definitions that the isomorphism $B_{\mathbb{T}}$ maps Λ^{\perp} onto $X^*(\mathbb{T})$.

Lemma 3.1. *The following statements hold.*

- (1) $Q_{\mathbb{T}}$ takes integral values on the lattice Λ .
- (2) $\ell Q_{\mathbb{T}}$ takes integral values on the lattice Λ^{\perp} .

Proof. This follows easily from Lemma I.2. □

Now, as we already mentioned, the quadratic form $Q_{\mathbb{T}}$ is invariant under $\Omega \times \Omega$. The lattice $\Lambda = X_*(\mathbb{T})$, however, is invariant only under $W \times W$ and the diagonal copy of Ω_0 . In any event, it makes sense to say that the quadratic form $Q_{\mathbb{T}}$ on Λ is invariant under $(W \times W) \rtimes \Omega_0$.

3.4. Values of $Q_{\mathbb{T}}$ on roots and coroots. The root lattice inside $X^*(\mathbb{T})$ can of course be identified with $X^*(\mathbb{T}_{\text{ad}})$. The root α then becomes the pair $(\alpha, 0)$ in the description (3.2) of $X^*(\mathbb{T})$. The situation for the coroot lattice is parallel, and the coroot α^\vee becomes the pair $(\alpha^\vee, 0)$ in the description (3.1) of $X_*(\mathbb{T})$.

Lemma 3.2. *There are equalities*

$$Q_{\mathbb{T}}(\alpha^\vee) = \ell(\alpha^\vee), \quad (\ell Q_{\mathbb{T}})(\alpha) = \ell(\alpha).$$

Proof. We have $Q_{\mathbb{T}}(\alpha^\vee) = Q_1(\alpha^\vee) = \ell(\alpha^\vee)$. Similarly, $(\ell Q_{\mathbb{T}})(\alpha) = (\ell Q_1)(\alpha) = \ell(\alpha)$. In the first chain of equalities, we used the definition of $\ell(\alpha^\vee)$, and in the second we used Lemma I.1(2). \square

Corollary 3.3. *When $\ell = 1$, we have $\alpha^\vee = \alpha$ and $Q_{\mathbb{T}}(\alpha^\vee) = Q_{\mathbb{T}}(\alpha) = 1$. When $\ell = 2, 3$, there are two root lengths, and we have*

- $Q_{\mathbb{T}}(\alpha^\vee) = 1$ when α is long, and
- $(\ell Q_{\mathbb{T}})(\alpha) = 1$ when α is short.

Proof. It follows from Lemma I.1(1) that $\alpha^\vee = \alpha$ when $\ell = 1$. The rest follows from Lemma 3.2. \square

3.5. The quadratic form $Q_{\mathbb{G}}$ on $\text{Lie}(\mathbb{G})$. In Appendix I we studied a certain quadratic form Q_2 on $\text{Lie}(\mathbb{G}_{\text{sc}}) = \text{Lie}(\mathbb{T}_{\text{sc}}) \oplus \mathbb{V}$. In this subsection we extend Q_2 to a quadratic form $Q_{\mathbb{G}}$ on $\text{Lie}(\mathbb{G}) = \text{Lie}(\mathbb{T}) \oplus \mathbb{V}$.

Let $Q_{\mathbb{G}}$ be the unique quadratic form on $\text{Lie}(\mathbb{G})$ such that

- $\text{Lie}(\mathbb{T})$ and \mathbb{V} are orthogonal,
- $Q_{\mathbb{G}}$ restricts to $Q_{\mathbb{T}}$ on $\text{Lie}(\mathbb{T})$, and
- $Q_{\mathbb{G}}$ restricts to Q_2 on \mathbb{V} .

It is clear that $Q_{\mathbb{G}}$ restricts to Q_2 on all of $\text{Lie}(\mathbb{G}_{\text{sc}})$. It is also clear that $Q_{\mathbb{G}}$ is invariant under \mathbb{G}_{ad} . Now Q_2 is invariant under all automorphisms of \mathbb{G}_{sc} , and $Q_{\mathbb{T}}$ is invariant under Ω_0 (acting diagonally on $\mathfrak{a} \times \mathfrak{a}$). It follows that $Q_{\mathbb{G}}$ is invariant under the natural action of $\mathbb{G}_{\text{ad}} \rtimes \Omega_0$ on $\text{Lie}(\mathbb{G})$. (In greater detail, \mathbb{G}_{ad} is acting by the adjoint representation, and Ω_0 is acting by the action induced by the diagonal action on $\mathbb{G}_{\text{sc}} \times \mathbb{T}_{\text{sc}}$.)

In subsection I.4 a nondegenerate quadratic form $Q_{\mathbb{V}}$ is defined, and in subsection I.5 there is a discussion of the orthogonal direct sum decomposition

$$(3.3) \quad (\mathbb{V}, Q_{\mathbb{V}}) = (\mathbb{V}', Q_{\mathbb{V}'}) \oplus (\mathbb{V}'', Q_{\mathbb{V}''})$$

as well as the relationship between the nondegenerate quadratic forms $Q_{\mathbb{V}'}$, $Q_{\mathbb{V}''}$ and the ones we obtain by restriction from $Q_{\mathbb{G}}$. From this

discussion we conclude that $(\mathrm{Lie}(\mathbb{G}), Q_{\mathbb{G}})$ decomposes as the orthogonal direct sum

$$(3.4) \quad (\mathrm{Lie}(\mathbb{G}), Q_{\mathbb{G}}) = (\mathrm{Lie}(\mathbb{T}), Q_{\mathbb{T}}) \oplus (\mathbb{V}', Q_{\mathbb{V}'}) \oplus (\mathbb{V}'', \ell Q_{\mathbb{V}''}).$$

(The direct summand \mathbb{V}'' is zero in the simply laced case.)

4. THE SPLIT F -GROUP \mathbf{G} AND THE QUADRATIC FORM $Q_{\mathbf{G}}$ ON ITS LIE ALGEBRA

4.1. The pinned F -group \mathbf{G} . We are almost done with our preliminary analysis of the situation over \mathbb{Z} . We extend scalars from \mathbb{Z} to F . In this way we obtain a pinned F -group $(\mathbf{G}, \mathbf{B}, \mathbf{T}, \{\eta_{\alpha}\})$.

4.2. The quadratic forms $Q_{\mathbf{G}}, Q_{\mathbf{T}}, Q_{\mathbf{V}}, Q_{\mathbf{V}'}, Q_{\mathbf{V}''}$. By extension of scalars the quadratic form $Q_{\mathbb{G}}$ yields a quadratic form $Q_{\mathbf{G}}$ on $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$. Similarly, we obtain from $Q_{\mathbb{T}}$ a quadratic form $Q_{\mathbf{T}}$ on $\mathfrak{t} := \mathrm{Lie}(\mathbf{T})$, and from (3.3) we obtain $(\mathbf{V}, Q_{\mathbf{V}}) = (\mathbf{V}', Q_{\mathbf{V}'}) \oplus (\mathbf{V}'', Q_{\mathbf{V}''})$.

The quadratic forms $Q_{\mathbf{V}}, Q_{\mathbf{V}'}, Q_{\mathbf{V}''}$ are always nondegenerate. We will observe later that, when ℓ is invertible in F , the quadratic forms $Q_{\mathbf{G}}, Q_{\mathbf{T}}$ are also nondegenerate. When ℓ is zero in F , we will need substitutes for $Q_{\mathbf{G}}, Q_{\mathbf{T}}$. We discuss these next.

4.3. The quadratic forms $Q_{\mathfrak{g}'}, Q_{\mathfrak{g}''}, Q_{\mathfrak{t}'}, Q_{\mathfrak{t}''}$. In this subsection we assume that ℓ is zero in F . Then $Q_{\mathbf{G}}$ is degenerate and is therefore not useful for proving the main theorem. We get around this difficulty by applying Lemma J.1 to $(\mathrm{Lie}(\mathbb{G}), Q_{\mathbb{G}})$ and the three direct summands appearing in (3.4). For each of these four lattices Λ , Lemma J.1 provides a short exact sequence

$$(4.1) \quad 0 \rightarrow \Lambda^{\perp}/\Lambda \xrightarrow{\ell} \Lambda/\ell\Lambda \rightarrow \Lambda/\ell\Lambda^{\perp} \rightarrow 0$$

as well as nondegenerate quadratic forms on the \mathbb{F}_{ℓ} -vector spaces $\Lambda/\ell\Lambda^{\perp}$ and Λ^{\perp}/Λ .

Now our assumption that ℓ is zero in F means that \mathbb{F}_{ℓ} is the prime field in F . So we may tensor (4.1) over \mathbb{F}_{ℓ} with F , obtaining short exact sequences of F -vector spaces

$$0 \rightarrow \bar{\Lambda}'' \rightarrow \Lambda \otimes_{\mathbb{F}_{\ell}} F \rightarrow \bar{\Lambda}' \rightarrow 0,$$

as well as canonical nondegenerate quadratic forms on $\bar{\Lambda}'$ and $\bar{\Lambda}''$.

We need notation for all these objects. When $\Lambda = \mathrm{Lie}(\mathbb{G})$, we write

$$(4.2) \quad 0 \rightarrow \mathfrak{g}'' \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}' \rightarrow 0$$

for the vector spaces and $Q_{\mathfrak{g}'}, Q_{\mathfrak{g}''}$ for the quadratic forms. Of course \mathfrak{g} is the Lie algebra of \mathbf{G} . Observe that (4.2) is actually a short exact

sequence of \mathbf{G}_{ad} -modules, and that $Q_{\mathbf{g}''}, Q_{\mathbf{g}'}$ are both invariant under \mathbf{G}_{ad} .

From the remaining three lattices $\text{Lie}(\mathbb{T}), \mathbb{V}', \mathbb{V}''$ we obtain short exact sequences of $N_{\mathbf{G}}(\mathbb{T})$ -modules

$$(4.3) \quad \begin{aligned} 0 &\rightarrow \mathbf{t}'' \rightarrow \mathbf{t} \rightarrow \mathbf{t}' \rightarrow 0 \\ 0 &\rightarrow 0 \rightarrow \mathbf{V}' \xrightarrow{\text{id}} \mathbf{V}' \rightarrow 0 \\ 0 &\rightarrow \mathbf{V}'' \xrightarrow{\text{id}} \mathbf{V}'' \rightarrow 0 \rightarrow 0 \end{aligned}$$

as well as quadratic forms $Q_{\mathbf{t}'}, Q_{\mathbf{t}''}, Q_{\mathbf{V}'}, Q_{\mathbf{V}''}$. It needs to be stressed here that the nondegenerate quadratic forms on \mathbf{V}' and \mathbf{V}'' produced by Lemma J.1 really do agree with the forms $Q_{\mathbf{V}'}, Q_{\mathbf{V}''}$ discussed in the previous subsection. In other words, the factor of ℓ appearing in the last summand in (3.4) has disappeared.

4.4. Spinor norm on the Weyl group. We are done constructing all the quadratic forms we need, and are almost ready to complete the proof of the main theorem. Before doing so, we are going to calculate some spinor norms (see section G.3 for a review of the spinor norm map δ) for elements in the Weyl group, thereby paving the way for applying Fröhlich's theorem. We begin by defining a sign character on W .

4.4.1. Definition of ε'' . On the Weyl group we have the usual sign character $\varepsilon : W \rightarrow \{\pm 1\}$. It takes the value -1 on every reflection $w_\alpha = w_{\alpha^\vee} \in W$. More useful for us, however, is another sign character ε'' on W , defined as follows.

When $\ell = 1$ (the simply laced case in which there is only one root length), we take ε'' to be ε . When $\ell \neq 1$, so that there are two root lengths, we take ε'' to be the unique sign character such that

$$\varepsilon''(w_\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is long,} \\ -1 & \text{if } \alpha \text{ is short.} \end{cases}$$

We will mainly need ε'' , but there will be one occasion when we will also need the sign character ε' defined through the equality

$$\varepsilon = \varepsilon' \varepsilon''.$$

We will need it only when there are two root lengths, in which case

$$\varepsilon'(w_\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is short,} \\ -1 & \text{if } \alpha \text{ is long.} \end{cases}$$

We will sometimes view ε' and ε'' as having values in the additive group \mathbb{F}_2 , hence as elements in $H^1(W, \mathbb{F}_2)$.

4.4.2. *Spinor norms for W when ℓ is invertible in F .* Assume that ℓ is invertible in F . We work with the split F -group \mathbf{G} . We are particularly interested in the nondegenerate quadratic space $(\mathbf{t}, Q_{\mathbf{T}})$. The action of the Weyl group preserves $Q_{\mathbf{T}}$ and therefore yields a canonical homomorphism

$$(4.4) \quad W \rightarrow O(Q_{\mathbf{T}})(F).$$

For any root α the coroot H_{α} is a vector in \mathbf{t} , and the homomorphism (4.4) maps $w_{\alpha} = w_{\alpha^{\vee}}$ to the reflection $r_{H_{\alpha}}$ in that vector.

Composing the homomorphism (4.4) with the spinor norm homomorphism

$$O(Q_{\mathbf{T}})(F) \xrightarrow{\delta} F^{\times}/(F^{\times})^2,$$

we obtain a homomorphism

$$(4.5) \quad W \rightarrow F^{\times}/(F^{\times})^2.$$

Lemma 4.1. *The value of the homomorphism (4.5) on $w \in W$ is equal to the square class of*

$$\begin{cases} 1 & \text{if } \varepsilon''(w) = 1 \\ \ell & \text{if } \varepsilon''(w) = -1 \end{cases}$$

In other words the homomorphism (4.5) is equal to the element

$$\varepsilon'' \otimes \ell \in H^1(W, \mathbb{F}_2) \otimes F^{\times}/(F^{\times})^2 = \text{Hom}(W, F^{\times}/(F^{\times})^2).$$

Proof. It is enough to prove this when w is a reflection $w_{\alpha} = w_{\alpha^{\vee}}$. Then w is sent to reflection in the coroot H_{α} , whose spinor norm (see subsection G.5) is the square class of $Q_{\mathbf{T}}(H_{\alpha}) = \ell(\alpha^{\vee}) \in F^{\times}$. \square

4.4.3. *Spinor norms for W when ℓ is zero in F .* Now the Weyl group W acts as a group of automorphisms of the nondegenerate quadratic spaces $(\mathbf{t}', Q_{\mathbf{t}'})$, $(\mathbf{t}'', Q_{\mathbf{t}''})$, so there are natural homomorphisms from W to $O(Q_{\mathbf{t}'})(F)$ and $O(Q_{\mathbf{t}''})(F)$. As usual we denote spinor norm maps by δ .

Lemma 4.2. *The composed maps*

$$(4.6) \quad W \rightarrow O(Q_{\mathbf{t}'})(F) \xrightarrow{\delta} F^{\times}/(F^{\times})^2$$

$$(4.7) \quad W \rightarrow O(Q_{\mathbf{t}''})(F) \xrightarrow{\delta} F^{\times}/(F^{\times})^2$$

are both trivial.

Proof. Let $w \in W$ and write w' (resp., w'') for the image of w in $O(Q_{\mathbf{t}'})$ (resp., $O(Q_{\mathbf{t}''})$). It is enough to prove that the spinor norms of w' and w'' are trivial when w is a reflection. So let us consider the reflection $w = w_{\alpha} = w_{\alpha^{\vee}} \in W$. From Corollary 3.3 we know that

- $Q_{\mathbb{T}}(\alpha^{\vee}) = 1$ when α is long, and
- $(\ell Q_{\mathbb{T}})(\alpha) = 1$ when α is short.

So, when α is long (resp., short), Lemma J.2 implies that (w', w'') is equal to $(r_{v'}, 1)$ (resp., $(1, r_{v''})$), where v' is the image of α^{\vee} in $\Lambda/\ell\Lambda^{\perp}$ (resp., v'' is the image of α in Λ^{\perp}/Λ). Now $Q_{\mathbf{t}'}(v') = Q_{\mathbb{T}}(\alpha^{\vee}) = 1$ (resp., $Q_{\mathbf{t}''}(v'') = (\ell Q_{\mathbb{T}})(\alpha) = 1$), so (see subsection G.5) all the relevant spinor norms are indeed trivial. \square

5. END OF THE PROOF OF THE MAIN THEOREM

5.1. The quasi-split F -group G and the quadratic form Q_G . The Galois group acts trivially on Ω_0 , so 1-cocycles of Γ in Ω_0 are simply homomorphisms $\varphi_0 : \Gamma \rightarrow \Omega_0$. As before we are interested in the diagonal action of Ω_0 on $\mathbf{G}_{\text{sc}} \times \mathbf{T}_{\text{sc}}$ and the induced action on the quotient \mathbf{G} . We use this action, together with φ_0 , to twist $(\mathbf{G}, \mathbf{B}, \mathbf{T}, \{\eta_{\alpha}\})$. The result is a quasi-split F -group G equipped with an F -splitting $(B_0, T_0, \{\eta_{\alpha}\})$. (One has $\sigma(\eta_{\alpha}) = \eta_{\sigma\alpha}$ for all $\sigma \in \Gamma$.) We are writing T_0 for the twist of \mathbf{T} by φ_0 in order to keep T in reserve as notation for an arbitrary maximal F -torus in G . The F -torus T_0 is of course very special, because there exists a Borel F -subgroup, namely B_0 , containing it.

Now the action of Ω_0 preserves the quadratic form $Q_{\mathbf{G}}$, which therefore remains F -rational after twisting. In other words we may now view it as a canonically defined quadratic form, call it Q_G , on $\mathfrak{g} = \text{Lie}(G)$. See subsection G.6 for a review of twisting of quadratic forms.

It follows from Lemmas 3.1 and J.1, together with our comparison of Q_2 and $Q_{\mathbb{V}}$ (see subsection I.3) that Q_G is nondegenerate when ℓ is invertible in F .

5.2. What remains to be proved? For any two maximal F -tori T, \tilde{T} in G we need to prove that

$$\gamma(Q_{\tilde{V}}, \psi) \gamma(Q_V, \psi)^{-1} = \epsilon_L(X_*(\tilde{T})_{\mathbb{C}} - X_*(T)_{\mathbb{C}}, \psi).$$

Applying Corollary F.2, we are reduced to proving that

$$(5.1) \quad HW(Q_V, Q_{\tilde{V}}) = \overline{SW}(X_*(\tilde{T}) - X_*(T)).$$

(The expressions HW and \overline{SW} appearing here are discussed in the appendices.)

The maximal torus T is obtained from T_0 by twisting by a 1-cocycle in the normalizer of T_0 in G . So T is obtained from \mathbf{T} by twisting by a 1-cocycle of the form $n_{\sigma}\varphi_0(\sigma)$ with n_{σ} in the \overline{F} -points of the normalizer $N_{\mathbf{G}}(\mathbf{T})$ of \mathbf{T} in \mathbf{G} . The same goes for \tilde{T} , which is obtained from \mathbf{T} by twisting by a 1-cocycle of the form $\tilde{n}_{\sigma}\varphi_0(\sigma)$.

At this point we need to consider two cases: either ℓ is invertible in F , or ℓ is zero in F .

5.3. End of the proof when ℓ is invertible in F . We now assume that ℓ is invertible in F , which ensures that Q_G is nondegenerate. We write Q_T for the restriction of Q_G to $\mathfrak{t} = \text{Lie}(T)$; it is obtained by twisting $Q_{\mathbf{T}}$. In fact the whole situation $(\mathfrak{g}, Q_G) = (\mathfrak{t}, Q_T) \oplus (V, Q_G|_V)$ is obtained by twisting $(\mathfrak{g}, Q_{\mathbf{G}}) = (\mathfrak{t}, Q_{\mathbf{T}}) \oplus (\mathbf{V}, Q_{\mathbf{G}}|_{\mathbf{V}})$ by the 1-cocycle $n_\sigma \varphi_0(\sigma)$. (Recall from subsection 4.2 that \mathbf{V} is obtained by extension of scalars from \mathbb{V} .)

We let w_σ denote the image of n_σ under the natural homomorphism $N_{\mathbf{G}}(\mathbf{T}) \rightarrow W$. Now put $\varphi(\sigma) := w_\sigma \varphi_0(\sigma) \in W \rtimes \Omega_0 = \Omega$. Then φ is a homomorphism $\Gamma \rightarrow \Omega$, and it is also true that T is obtained from the split torus \mathbf{T} by twisting by φ , viewed as a 1-cocycle of Γ in Ω , with Γ acting trivially on Ω . Similarly we obtain $\tilde{w}_\sigma, \tilde{\varphi}$ from \tilde{T} . In addition to $\varphi, \tilde{\varphi}$ we will soon need the sign character χ on Γ defined by

$$(5.2) \quad \chi(\sigma) := \varepsilon''(\tilde{w}_\sigma w_\sigma^{-1}) = \varepsilon''(\tilde{\varphi}(\sigma) \varphi(\sigma)^{-1}).$$

To see that χ really is multiplicative in σ , one needs to notice that the sign character ε'' (defined in subsection 4.4.1) is invariant under the action of Ω_0 on W .

Twisting the decomposition $\mathbf{V} = \mathbf{V}' \oplus \mathbf{V}''$, we obtain a decomposition $V = V' \oplus V''$. When $\ell = 1$ (the simply laced case) $V' = V$ and $V'' = 0$. When $\ell = 2, 3$ (the non-simply laced case) V' is the sum of the root spaces for the long roots of T , and V'' is the sum of the root spaces for the short roots of T . The twist of $Q_{\mathbf{V}}$ coincides with the quadratic form Q_V defined in subsection 1.3.3. We denote the restriction of Q_V to V' (resp., V'') by $Q_{V'}$ (resp., $Q_{V''}$).

It follows from (3.4) that

$$(5.3) \quad (\mathfrak{g}, Q_G) = (\mathfrak{t}, Q_T) \oplus (V', Q_{V'}) \oplus (V'', \ell Q_{V''}).$$

All these quadratic spaces are nondegenerate. From this orthogonal decomposition (and its analog for \tilde{T}), we find that

$$\text{Wall}(Q_T) + \text{Wall}(Q_{V'}) + \text{Wall}(\ell Q_{V''})$$

is equal to its analog for \tilde{T} (see Appendix A for a discussion of the Wall homomorphism). It follows from Lemma C.2(2) that

$$\begin{aligned} 0 &= HW(Q_T, Q_{\tilde{T}}) + HW(Q_{V'}, Q_{\tilde{V}'}) + HW(\ell Q_{V''}, \ell Q_{\tilde{V}''}) \\ &= HW(Q_T, Q_{\tilde{T}}) + HW(Q_V, Q_{\tilde{V}}) + HW(\ell Q_{V''}, \ell Q_{\tilde{V}''}) - HW(Q_{V''}, Q_{\tilde{V}''}). \end{aligned}$$

To finish the proof of (5.1), it remains only to justify the following claims.

$$(1) \quad HW(Q_T, Q_{\tilde{T}}) = \overline{SW}(\varphi) - \overline{SW}(\tilde{\varphi}) + \xi(\chi \otimes \ell).$$

- (2) $HW(\ell Q_{V''}, \ell Q_{\tilde{V}''}) = HW(Q_{V''}, Q_{\tilde{V}''}) + \xi(\chi \otimes \ell)$.
- (3) $\overline{SW}(X_*(T)_{\mathbb{C}}) = \overline{SW}(\varphi)$, and similarly for $(\tilde{T}, \tilde{\varphi})$.

Here χ is the sign character on W we defined (see equation (5.2)) using ε'' , φ and $\tilde{\varphi}$. The homomorphism ξ is defined in Appendix G.

For item (1) we appeal to the result of Fröhlich which is reviewed (and extended to characteristic 2) in subsection G.4. We apply his result to φ , viewed as an orthogonal representation of Γ on the quadratic space $(\mathbf{t}, Q_{\mathbf{T}})$ over F . Fröhlich's result says that

$$HW(Q_T, Q_{\mathbf{T}}) = \overline{SW}(\varphi) + \xi(\delta \circ \varphi),$$

and similarly with (T, φ) replaced by $(\tilde{T}, \tilde{\varphi})$. It follows that

$$\begin{aligned} HW(Q_T, Q_{\tilde{T}}) &= HW(Q_T, Q_{\mathbf{T}}) + HW(Q_{\mathbf{T}}, Q_{\tilde{T}}) \\ &= \overline{SW}(\varphi) + \xi(\delta \circ \varphi) - \overline{SW}(\tilde{\varphi}) - \xi(\delta \circ \tilde{\varphi}) \\ &= \overline{SW}(\varphi) - \overline{SW}(\tilde{\varphi}) + \xi(\chi \otimes \ell). \end{aligned}$$

The last equality follows from Lemma 4.1.

For item (2) we apply Lemma A.6, bearing in mind equation (G.3), thereby obtaining

$$\begin{aligned} \text{Wall}(\ell Q_{V''}) &= \text{Wall}(Q_{V''}) + \xi(\chi_{Q_{V''}} \otimes \ell), \\ \text{Wall}(\ell Q_{\tilde{V}''}) &= \text{Wall}(Q_{\tilde{V}''}) + \xi(\chi_{Q_{\tilde{V}''}} \otimes \ell), \end{aligned}$$

from which it follows that

$$HW(\ell Q_{V''}, \ell Q_{\tilde{V}''}) = HW(Q_{V''}, Q_{\tilde{V}''}) + \xi(\chi_{Q_{V''}}^{-1} \chi_{Q_{\tilde{V}''}} \otimes \ell)$$

So we just need to show that $\chi_{Q_{\tilde{V}''}}$ is the product of $\chi_{Q_{V''}}$ and χ .

For this it is enough to prove that the following square commutes:

$$\begin{array}{ccc} N_{\mathbb{G}}(\mathbb{T}) & \longrightarrow & O(Q_{\mathbb{V}''}) \\ \downarrow & & \deg \downarrow \\ W & \xrightarrow{\varepsilon''} & \mathbb{F}_2. \end{array}$$

Let's call the top horizontal arrow t . It is clear that the composed map $\deg \circ t$ is trivial on the identity component \mathbb{T} , so we are really trying to prove the equality of two sign characters on W . For this it suffices to show that they agree on every simple reflection. So, let α be a simple root, and consider $n_{\alpha} \in N_{\mathbb{G}}(\mathbb{T})$ whose image in W is the simple reflection s_{α} . We need to show that $\deg(t(n_{\alpha}))$ is trivial (resp., nontrivial) if α is long (resp., short).

Recall that \mathbb{V}'' is the direct sum of the root spaces for all of the short roots. Now the set R'' of short roots is the disjoint union of the sets

$R'' \cap R^+$ and $R'' \cap R^-$, so \mathbb{V}'' decomposes accordingly as

$$(5.4) \quad \mathbb{V}'' = \mathbb{V}''^+ \oplus \mathbb{V}''^-.$$

The action of s_α on R interchanges α , $-\alpha$. It also preserves the sets $R^+ \setminus \{\alpha\}$ and $R^- \setminus \{-\alpha\}$, as well as the set R'' .

So, when α is long, s_α preserves both $R'' \cap R^+$ and $R'' \cap R^-$, and therefore the orthogonal transformation $t(n_\alpha)$ preserves the decomposition (5.4) of \mathbb{V}'' as the orthogonal direct sum of two isotropic subspaces. It follows that $\deg(t(n_\alpha))$ is trivial, as desired.

When α is short, similar considerations imply that $\deg(t(n_\alpha)) = \deg(t_{\pm\alpha}(n_\alpha))$, where $t_{\pm\alpha}(n_\alpha)$ denotes the image of n_α in the orthogonal group of the hyperbolic plane $\mathrm{Lie}(\mathbb{G}_\alpha) \oplus \mathrm{Lie}(\mathbb{G}_{-\alpha})$. But n_α interchanges these two root spaces, so the sign character \deg for the orthogonal group of this hyperbolic plane is nontrivial on $t_{\pm\alpha}(n_\alpha)$, as desired. The verification of item (2) is now complete.

Finally, for item (3) we appeal to Appendix K. We are now done with the case in which ℓ is invertible in F .

5.4. The case when ℓ is zero in F . Now assume that ℓ is 0 in F , which is to say that the prime field of F is \mathbb{F}_ℓ . The canonical quadratic form Q_G is then degenerate, so it does not have a Weil index, and we must modify the procedure used when ℓ is invertible in F .

Since the hypothesis that $\ell = 0$ in F implies that ℓ is either 2 or 3, the irreducible root system R under consideration is not simply laced, and the group Ω_0 is necessarily trivial. So no twisting takes place, and G coincides with the split F -group \mathbf{G} obtained from \mathbb{G} by extension of scalars from \mathbb{Z} to F . Moreover T_0 , $\mathrm{Lie}(T_0)$, \mathfrak{g} coincide with \mathbf{T} , \mathfrak{t} , \mathfrak{g} respectively.

The maximal torus T is obtained from \mathbf{T} by twisting by a 1-cocycle n_σ in $N_{\mathbf{G}}(\mathbf{T})$. As before we denote by w_σ the image of n_σ under $N_{\mathbf{G}}(\mathbf{T}) \rightarrow W$. Because \mathbf{T} is split, the map $\varphi : \Gamma \rightarrow W$ defined by $\varphi(\sigma) = w_\sigma$ is a homomorphism.

Twisting the short exact sequences (4.3) by the 1-cocycle n_σ then yields short exact sequences

$$(5.5) \quad \begin{aligned} 0 &\rightarrow \mathfrak{t}'' \rightarrow \mathfrak{t} \rightarrow \mathfrak{t}' \rightarrow 0 \\ 0 &\rightarrow 0 \rightarrow V' \xrightarrow{\mathrm{id}} V' \rightarrow 0 \\ 0 &\rightarrow V'' \xrightarrow{\mathrm{id}} V'' \rightarrow 0 \rightarrow 0 \end{aligned}$$

as well as quadratic forms $Q_{V'}$, $Q_{V''}$, $Q_{V'}$, $Q_{V''}$. Moreover $Q_{V'}$ and $Q_{V''}$ coincide with the quadratic forms obtained by restriction from Q_V on $V = V' \oplus V''$ (see the remark at the end of subsection 4.3).

Observe that there are T -invariant orthogonal direct sum decompositions

$$\mathfrak{g}' = \mathfrak{t}' \oplus V', \quad \mathfrak{g}'' = \mathfrak{t}'' \oplus V''.$$

To finish the proof we are going to use $(\mathfrak{g}' \oplus \mathfrak{g}'', Q_{\mathfrak{g}'} \oplus Q_{\mathfrak{g}''})$ in the same way we used (\mathfrak{g}, Q_G) before. Observe that

$$(5.6) \quad Q_{\mathfrak{g}'} \oplus Q_{\mathfrak{g}''} = Q_{\mathfrak{t}'} \oplus Q_{\mathfrak{t}''} \oplus Q_V.$$

So, although the notation is more elaborate, the situation is simpler in one respect: the restriction of $Q_{\mathfrak{g}'} \oplus Q_{\mathfrak{g}''}$ to $V' \oplus V'' = V$ is precisely the canonical quadratic form Q_V we need to understand.

Everything we have just done for T can also be done for \tilde{T} . Objects for \tilde{T} are always denoted by adding a tilde to the name of the corresponding object for T . From (5.6) and its analog for \tilde{T} we find that

$$0 = HW(Q_{\mathfrak{t}'}, Q_{\tilde{\mathfrak{t}'}}) + HW(Q_{\mathfrak{t}''}, Q_{\tilde{\mathfrak{t}''}}) + HW(Q_V, Q_{\tilde{V}}).$$

To complete the proof of (5.1) it suffices to verify the following claims.

- (1) $HW(Q_{\mathfrak{t}'}, Q_{\tilde{\mathfrak{t}'}}) = \overline{SW}(\varphi') - \overline{SW}(\tilde{\varphi}')$.
- (2) $HW(Q_{\mathfrak{t}''}, Q_{\tilde{\mathfrak{t}''}}) = \overline{SW}(\varphi'') - \overline{SW}(\tilde{\varphi}'')$.
- (3) $\overline{SW}(X_*(T)_{\mathbb{C}}) = \overline{SW}(\varphi') + \overline{SW}(\varphi'')$.

Here φ' (resp., φ'') denotes the orthogonal representation of Γ on \mathfrak{t}' (resp., \mathfrak{t}'') obtained from $\varphi : \Gamma \rightarrow W$ and the homomorphism $W \rightarrow O(Q_{\mathfrak{t}'})$ (resp., $W \rightarrow O(Q_{\mathfrak{t}''})$).

To prove (1) we observe that

$$\begin{aligned} HW(Q_{\mathfrak{t}'}, Q_{\tilde{\mathfrak{t}'}}) &= HW(Q_{\mathfrak{t}'}, Q_{\mathfrak{t}'}) - HW(Q_{\tilde{\mathfrak{t}'}, Q_{\mathfrak{t}'}} \\ &= \overline{SW}(\varphi') - \overline{SW}(\tilde{\varphi}'). \end{aligned}$$

Here we applied Fröhlich's result to φ' , $\tilde{\varphi}'$, using Lemma 4.2, which asserts that the spinor norm map is trivial on the image of W in $O(Q_{\mathfrak{t}'})$. Claim (2) is proved in the same way.

To prove (3) we need to consider the subcases $\ell = 2$ and $\ell = 3$ separately. When $\ell = 2$, our field F has characteristic 2, and the only real content in (3) is that $\overline{SW}_1(X_*(T)_{\mathbb{C}}) = \overline{SW}_1(\varphi') + \overline{SW}_1(\varphi'')$. This is not quite as obvious as one might think, since the sign character \deg on orthogonal groups in characteristic 2 is not given by the determinant. Fortunately the Weyl group is generated by reflections, so Lemma J.2 does the job. (In characteristic 2 it remains true that the sign character \deg is nontrivial on reflections.)

It remains to prove (3) when $\ell = 3$. We have three orthogonal representations of W , one on the \mathbb{C} -vector space $X_*(T)_{\mathbb{C}}$, one on the

F -vector space \mathfrak{t}' , and one on the F -vector space \mathfrak{t}'' . It suffices to prove the equality

$$SW(X_*(\mathbb{T})_{\mathbb{C}}) = SW(\mathfrak{t}') + SW(\mathfrak{t}'')$$

of elements in the group $1 + H^1(W, \mathbb{F}_2) + H^2(W, \mathbb{F}_2)$.

Since $\ell = 3$, our root system is G_2 . Therefore $\mathbb{G}_{\text{sc}} = \mathbb{G}_{\text{ad}}$, and \mathbb{T} is simply the cartesian product of two copies of \mathbb{T}_{sc} . The second copy (the one that is central in \mathbb{G}) has no effect on anything, so we are reduced to proving the equality

$$SW(X_*(\mathbb{T}_{\text{sc}})_{\mathbb{C}}) = SW(\mathfrak{u}') + SW(\mathfrak{u}'') \in 1 + H^1(W, \mathbb{F}_2) + H^2(W, \mathbb{F}_2),$$

where \mathfrak{u}'' is the line in $\mathfrak{u} := X_*(\mathbb{T}_{\text{sc}})_F$ generated by any long coroot, and \mathfrak{u}' is the quotient line $\mathfrak{u}/\mathfrak{u}''$. (Remember that F has characteristic 3. The reflection representation \mathfrak{u} of W is reducible, but not semisimple. The unique W -invariant line is \mathfrak{u}'' ; it contains all the long coroots.)

Recall from subsection 4.4.1 that the sign character ε on W is the product of two sign characters ε' and ε'' . One checks that W acts on the line \mathfrak{u}' (resp., \mathfrak{u}'') by the sign character ε' (resp., ε'').

The Weyl group is dihedral of order 12. It is not difficult to see that

$$SW(X_*(\mathbb{T}_{\text{sc}})_{\mathbb{C}}) = 1 + \varepsilon + \varepsilon' \cup \varepsilon'' \in 1 + H^1(W, \mathbb{F}_2) + H^2(W, \mathbb{F}_2),$$

and this is the sum of $SW(\mathfrak{u}') = 1 + \varepsilon'$ and $SW(\mathfrak{u}'') = 1 + \varepsilon''$, as desired. The group law used here is the usual one:

$$(1 + a + b)(1 + a' + b') = 1 + (a + a') + (a \cup a' + b + b').$$

APPENDIX A. WITT GROUP AND THE WALL HOMOMORPHISM

In this section we work over an arbitrary field F . By algebra we mean F -algebra, and by dimension we mean F -dimension. We choose a separable closure \overline{F} of F and write Γ for the Galois group $\text{Gal}(\overline{F}/F)$. Throughout Appendix A all quadratic spaces we consider are tacitly assumed to be nondegenerate, as these are the only ones relevant for the Witt group.

A.1. Quadratic étale algebras. Recall that an *étale* algebra is product of finitely many finite separable field extensions of F . A *quadratic étale* algebra is an étale algebra of dimension 2. A quadratic étale algebra E is either a separable quadratic field extension or else isomorphic to $F \times F$. In either case there is a unique nontrivial automorphism of E over F , denoted by $y \mapsto \bar{y}$. There is a canonical quadratic form Q_E on the 2-dimensional F -vector space E , given by $Q_E(y) = y\bar{y}$. Of course Q_E is nothing but the norm mapping $N_{E/F} : E \rightarrow F$.

Attached to E is a canonical sign character χ_E on Γ . When E is isomorphic to $F \times F$, the character χ_E is trivial. When E/F is a separable quadratic field extension, χ_E is the composed map

$$\Gamma \twoheadrightarrow \text{Gal}(E/F) \simeq \{\pm 1\}$$

(which is of course independent of the choice of embedding of E in \overline{F}).

A.2. Quaternion algebras. A *quaternion* algebra D is a central simple algebra of dimension 4. There are two possibilities. One is that D is a division algebra. The other is that D is *split*, i.e. isomorphic to $M_2 F$. In any case there is a canonical anti-involution $y \mapsto \bar{y}$ on D . It is characterized by the fact that $y + \bar{y}$ (resp., $y\bar{y}$) is the reduced trace (resp., reduced norm) of y . There is a canonical quadratic form Q_D on the 4-dimensional vector space D , given by $Q_D(y) = y\bar{y} = \bar{y}y$.

A.3. The quaternion algebras $D(E, a)$. There is a standard way of attaching a $\mathbb{Z}/2\mathbb{Z}$ -graded quaternion algebra $D = D(E, a)$ to a pair consisting of a quadratic étale algebra E and an element $a \in F^\times$. From now on we will write \mathbb{F}_2 rather than $\mathbb{Z}/2\mathbb{Z}$. The isomorphism class of $D(E, a)$ (as \mathbb{F}_2 -graded algebra) depends only on a modulo norms from E^\times .

Here is a description of $D = D_0 \oplus D_1$. As before we write $e \mapsto \bar{e}$ for the unique nontrivial automorphism of E over F .

- (1) The even part D_0 of D is E . So E is a subalgebra of D .
- (2) The odd part D_1 of D is free of rank 1 as both left and right D_0 -module, hence is 2-dimensional over F .
- (3) There is an element $x \in D_1$ such that
 - (a) $x^2 = a \in F \subset D_0$,
 - (b) $xe = \bar{e}x$ for all $e \in E = D_0$.

Observe that x is a basis for D_1 as both left and right D_0 -module.

The canonical anti-involution on D preserves the \mathbb{F}_2 -grading. It

- (1) acts on $D_0 = E$ by the unique nontrivial automorphism of E/F ,
and
- (2) acts on $D_1 = Ex$ by multiplication by -1 .

Because of the first item, there is no conflict in using $y \mapsto \bar{y}$ to denote both the unique nontrivial automorphism of E/F and the canonical anti-involution of D/F .

A.4. Quadratic spaces. A quadratic space is a pair (V, Q) consisting of a finite dimensional vector space V and a quadratic form Q on V (i.e. Q is an element in the second symmetric power of V^*). We then obtain a symmetric bilinear form B on V defined by $Q(x + y) = Q(x) + B(x, y) + Q(y)$. Note that $B(x, x) = 2Q(x)$. Thus B is an

alternating form when F has characteristic 2. A quadratic space (V, Q) is said to be nondegenerate when the bilinear form B is nondegenerate. In characteristic 2 a nondegenerate quadratic space is necessarily even dimensional. Throughout this appendix we tacitly assume that the quadratic spaces we consider are nondegenerate.

A.5. Even Witt group $W_0(F)$. We write $W_0(F)$ for the Witt group of even-dimensional quadratic spaces over F , and we call $W_0(F)$ the *even Witt group* of F . Given an even dimensional quadratic space (V, Q) over F , we write $[Q]$ for the class of (V, Q) in the even Witt group. We remind the reader that the equations

$$\begin{aligned} [Q] + [-Q] &= 0, \\ [Q_{F \times F}] &= 0 \end{aligned}$$

hold in $W_0(F)$. (Notice that $Q_{F \times F}$ is a hyperbolic plane.)

Given an even dimensional quadratic space (V, Q) over F , we write $C(Q)$ for the Clifford algebra of Q . Then $C(Q)$ is a central simple algebra of dimension 2^n , where $n = \dim(V)$. Moreover $C(Q)$ is an \mathbb{F}_2 -graded algebra. It is a standard fact that

$$(A.1) \quad C(Q_1 \oplus Q_2) = C(Q_1) \otimes_s C(Q_2).$$

Here $Q_1 \oplus Q_2$ denotes the orthogonal direct sum of Q_1 and Q_2 , and \otimes_s denotes the \mathbb{F}_2 -graded tensor product of \mathbb{F}_2 -graded algebras.

Here are some standard examples. The isomorphisms in the proposition are isomorphisms of \mathbb{F}_2 -graded algebras.

Proposition A.1.

- (1) *Let E be a quadratic étale algebra and let $a \in F^\times$. Then $C(aQ_E)$ is isomorphic to $D(E, a)$.*
- (2) *Let D be any quaternion algebra. Then $C(Q_D)$ is isomorphic to the algebra $M_2(D)$ of 2×2 -matrices with entries in D , graded by taking the diagonal matrices as the even part, and the anti-diagonal matrices as the odd part.*

Proof. In this proof we will use the following way of recognizing the Clifford algebra $C(Q)$ obtained from an even-dimensional quadratic space (V, Q) . Let $A = A_0 \oplus A_1$ be an \mathbb{F}_2 -graded algebra having the same dimension as $C(Q)$. Then giving an isomorphism $\rho : C(Q) \rightarrow A$ of \mathbb{F}_2 -graded algebras is the same as giving a linear map $\phi : V \rightarrow A_1$ such that $(\phi(v))^2 = Q(v)$ for all $v \in V$. (The simplicity of the algebra $C(Q)$ guarantees that ρ is injective, hence an isomorphism.)

First we prove (1). We abbreviate $D(E, a)$ to D . Giving an isomorphism $\rho : C(aQ_E) \rightarrow D$ is the same as giving a linear map $\phi : E \rightarrow D_1$

such that $\phi(e)^2 = ae\bar{e}$. Taking $\phi : E \rightarrow D_1$ to be

$$e \mapsto ex \in Ex = D_1$$

does the job. Here we are using the description of $D(E, a)$ given earlier.

Now we prove (2). Identify $M_2(D)$ with the \mathbb{F}_2 -graded endomorphism algebra $\text{End}_D(M)$ of the \mathbb{F}_2 -graded right D -module $M = D \oplus D$, the first copy of D being given even degree, and the second copy being given odd degree. Giving ρ is then the same as giving an F -linear map $d \mapsto \phi_d$ from D to the odd part of $\text{End}_D(M)$, subject to the condition $(\phi_d)^2 = d\bar{d}$. Taking

$$\phi_d(x, y) := (\bar{d}y, dx) \quad (x, y \in D)$$

does the job. □

A related fact is

Proposition A.2. *Let E be a quadratic étale algebra and let $a \in F^\times$. Put $D = D(E, a)$. Then Q_D is the orthogonal direct sum of Q_E and $-aQ_E$. Consequently there is an equality*

$$[Q_D] = [Q_E] - [aQ_E]$$

in the even Witt group.

Proof. As before we have $D = D_0 \oplus D_1 = E \oplus Ex$. It is easy to see that the direct sum decomposition $D = D_0 \oplus D_1$ is orthogonal, and that D_0 (resp., D_1) is isomorphic as quadratic space to (E, Q_E) (resp., $(E, -aQ_E)$). □

A.6. Wall's little graded Brauer group. The reference for this subsection is Wall's paper [Wal64]. Wall's little graded Brauer group is an analog of the usual Brauer group $\text{Br}(F)$. It is based on the \mathbb{F}_2 -graded tensor product of \mathbb{F}_2 -graded central simple algebras, just as the usual Brauer group is based on the ordinary tensor product of central simple algebras. The word “little” refers to the fact that one only considers \mathbb{F}_2 -graded algebras whose underlying algebra is central simple in the usual sense.

Wall has determined the structure of the little graded Brauer group, which we denote by $\text{Br}_s(F)$. (We consistently use a subscript s to indicate “super” analogs of standard concepts.) He did this by attaching two invariants to an \mathbb{F}_2 -graded central simple algebra $A = A_0 \oplus A_1$. The first is the sign character χ_E associated to the quadratic étale algebra $E = E(A)$ defined as the center of A_0 . The second is the class $[A]$ of A in the ordinary Brauer group $\text{Br}(F)$. We will use additive notation for $\text{Br}(F) = H^2(F, \overline{F}^\times)$. Likewise, we use additive notation in $H^1(F, \mathbb{F}_2)$, which we identify with the group of (continuous) sign characters on Γ .

Proposition A.3 (Wall).

(1) *The map*

$$A \mapsto (\chi_{E(A)}, [A])$$

induces a bijection from $\text{Br}_s(F)$ to the Cartesian product

$$H^1(F, \mathbb{F}_2) \times \text{Br}(F).$$

(2) *Use the isomorphism in the first part to put the structure of abelian group on $H^1(F, \mathbb{F}_2) \times \text{Br}(F)$. The addition law is then given by*

$$(\chi, x) + (\chi', x') = (\chi + \chi', x + x' + \chi \cup \chi'),$$

where \cup is the cup-product pairing

$$H^1(F, \mathbb{F}_2) \otimes H^1(F, \mathbb{F}_2) \rightarrow H^2(F, \bar{F}^\times)$$

induced by the pairing $\mathbb{F}_2 \otimes \mathbb{F}_2 \rightarrow \bar{F}^\times$ given by $m \otimes n \mapsto (-1_F)^{mn}$.

Observe that the pairings occurring in this proposition are trivial when the characteristic of F is 2. In this case, the little graded Brauer group is isomorphic as abelian group to the cartesian product $H^1(F, \mathbb{F}_2) \times \text{Br}(F)$.

We use the bijection in the proposition to identify $\text{Br}_s(F)$ with $H^1(F, \mathbb{F}_2) \times \text{Br}(F)$ set-theoretically. Thus, given an \mathbb{F}_2 -graded central simple algebra A , we view the class $[A]_s$ of A in $\text{Br}_s(F)$ as being the pair $(\chi_{E(A)}, [A])$.

Notice that the cup-product $\chi \cup \chi'$ occurring in the formula for the addition law always lies in the subgroup $\text{Br}_2(F) := \{x \in \text{Br}(F) : 2x = 0\}$ of the Brauer group. Therefore $\text{Br}_2(F)_s := \{(\chi, x) \in H^1(F, \mathbb{F}_2) \times \text{Br}(F) : 2x = 0\}$ is a subgroup of the little graded Brauer group.

Remark A.4. Let $(\chi, x) \in \text{Br}_s(F)$. Then its inverse with respect to the group law is $(\chi, x + \chi \cup \chi)$.

A.7. Wall homomorphism. Equation (A.1) implies that $Q \mapsto [C(Q)]_s$ induces a homomorphism from the even Grothendieck-Witt group to the little graded Brauer group $\text{Br}_s(F)$. As an immediate consequence of Proposition A.1, one has

Proposition A.5.

(1) *For any quadratic étale algebra E and any $a \in F^\times$*

$$[C(aQ_E)]_s = (\chi_E, [D(E, a)]).$$

(2) *For any quaternion algebra D*

$$[C(Q_D)]_s = (0, [D]).$$

Here, in accordance with our convention of using additive notation, 0 stands for the trivial sign character on Γ . Taking $E = F \times F$ in the first part of the proposition, one sees that $[C(Q)]_s$ is trivial when Q is a hyperbolic plane. Therefore $Q \mapsto [C(Q)]_s$ induces a homomorphism

$$W_0(F) \rightarrow \text{Br}_s(F).$$

This homomorphism was studied by Wall; we will denote it by

$$\text{Wall} : W_0(F) \rightarrow \text{Br}_s(F).$$

Concretely, $\text{Wall}(Q)$ is a pair $(\text{Wall}_1(Q), \text{Wall}_2(Q))$ with $\text{Wall}_1(Q) \in H^1(F, \mathbb{F}_2)$ and $\text{Wall}_2(Q) \in \text{Br}(F)$. The image of the Wall homomorphism $W_0(F) \rightarrow \text{Br}_s(F)$ is equal to the subgroup $\text{Br}_2(F)_s$.

A.8. Wall(aQ) versus Wall(Q). We denote by $\{\cdot, \cdot\}$ the pairing

$$H^1(F, \mathbb{F}_2) \otimes (F^\times / (F^\times)^2) \rightarrow \text{Br}_2(F)$$

in Chapter 14 of [Ser62]. The class of $D(E, a)$ in the Brauer group is then $\{\chi_E, a\}$. The pairing $\{\cdot, \cdot\}$ is reviewed in greater detail in subsection G.1.

We have the following corollary of Proposition A.5 (1). In the corollary we view $\text{Br}_2(F)$ as a subgroup of $\text{Br}_2(F)_s$ via the injection $x \mapsto (0, x)$.

Lemma A.6. *For any even dimensional quadratic space (V, Q) and any $a \in F^\times$ there is an equality*

$$\text{Wall}(aQ) - \text{Wall}(Q) = \{\chi_Q, a\} \in \text{Br}_2(F),$$

where χ_Q is the sign character $\text{Wall}_1(Q)$. Concretely, this means that

$$\begin{aligned} \text{Wall}_1(aQ) &= \text{Wall}_1(Q), \\ \text{Wall}_2(aQ) &= \text{Wall}_2(Q) + \{\chi_Q, a\}. \end{aligned}$$

Proof. This follows from Proposition A.5 (1), because any even dimensional quadratic space is an orthogonal direct sum of ones of the form bQ_E (E quadratic étale, $b \in F^\times$). \square

APPENDIX B. REVIEW OF THE CLIFFORD GROUP $Cl(Q)$ OF Q

We consider a nondegenerate quadratic space (V, Q) over a field F . We write $C = C(Q)$ for the Clifford algebra of (V, Q) . It is a graded (i.e., \mathbb{F}_2 -graded) algebra, and contains V as a linear subspace of C_1 . There is a canonical anti-involution $x \mapsto x_t$ on C such that $(v_1 v_2 \dots v_m)_t := v_m \dots v_2 v_1$ for $v_1, v_2, \dots, v_m \in V$.

We write C^{\times_s} for the group of invertible homogeneous elements in C . For $x \in C^{\times_s}$ we define an automorphism $\text{Int}_s(x)$ of the graded algebra C by putting

$$\text{Int}_s(x)(y) := (-1)^{\deg(x)\deg(y)} xyx^{-1}$$

for all homogenous $y \in C$. There is a short exact sequence

$$1 \rightarrow F^\times \rightarrow C^{\times_s} \xrightarrow{\text{Int}_s} \text{Aut}_s(C) \rightarrow 1,$$

where $\text{Aut}_s(C)$ denotes the group of automorphisms of the graded algebra C .

By functoriality any element u in the orthogonal group $O(Q)$ induces an automorphism u_* of the graded algebra C , and u_* preserves the subspace V of C_1 . It is easy to see that $u \mapsto u_*$ yields an isomorphism

$$O(Q) \rightarrow \{\theta \in \text{Aut}_s(C) : \theta(V) = V\},$$

with inverse $\theta \mapsto \theta|_V$.

The Clifford group $Cl(Q)$ is defined to be $\{x \in C^{\times_s} : \text{Int}_s(x)(V) = V\}$. There is a canonical homomorphism $\rho : Cl(Q) \rightarrow O(Q)$; it sends $x \in Cl(Q)$ to $\text{Int}_s(x)|_V$. The homomorphism ρ is surjective, so there is an exact sequence

$$(B.1) \quad 1 \rightarrow \mathbb{G}_m \rightarrow Cl(Q) \xrightarrow{\rho} O(Q) \rightarrow 1$$

of algebraic groups over F . The restriction of $\deg : C^{\times_s} \rightarrow \mathbb{F}_2$ to $Cl(Q)$ is trivial on \mathbb{G}_m , and hence induces a homomorphism of algebraic groups

$$(B.2) \quad \deg : O(Q) \rightarrow \mathbb{F}_2,$$

whose kernel is the special orthogonal group $SO(Q)$. Recall that $Cl(Q)$ contains $\{v \in V : Q(v) \neq 0\}$, and that for v in this subset $\rho(v)$ is equal to reflection r_v in v .

There is a canonical homomorphism

$$(B.3) \quad N : Cl(Q) \rightarrow \mathbb{G}_m$$

defined by $N(x) = x_t x$, where $x \mapsto x_t$ is the anti-involution defined near the beginning of this subsection. The restriction of N to $\mathbb{G}_m \subset Cl(Q)$ is the squaring map.

APPENDIX C. WALL INVARIANTS VERSUS HASSE-WITT INVARIANTS

We work over an arbitrary field F , with separable closure \overline{F} and absolute Galois group Γ . In this appendix we again tacitly assume that all quadratic spaces we consider are nondegenerate.

C.1. Relative Hasse-Witt invariants. Let n be a positive integer. Consider two n -dimensional quadratic spaces (V, Q) and (V', Q') . The difference between them is measured by an element

$$\text{inv}(Q', Q) \in H^1(F, O(Q)).$$

One obtains a specific 1-cocycle representing $\text{inv}(Q', Q)$ by choosing an \overline{F} -isomorphism $u : (V, Q) \rightarrow (V', Q')$ and then putting $a_\sigma := u^{-1}\sigma(u)$.

We write

$$\partial : H^1(F, O(Q)) \rightarrow H^2(F, \mathbb{G}_m)$$

for the connecting map for the short exact sequence (B.1). Recall from equation (B.3) the homomorphism $N : Cl(Q) \rightarrow \mathbb{G}_m$, which restricts to the squaring map on $\mathbb{G}_m \subset Cl(Q)$. Using it, one shows easily that the image of ∂ is contained in the subgroup $\text{Br}_2(F)$ of the Brauer group.

We write

$$\deg_* : H^1(F, O(Q)) \rightarrow H^1(F, \mathbb{F}_2)$$

for the map induced by the homomorphism $\deg : O(Q) \rightarrow \mathbb{F}_2$.

Definition C.1. The relative Hasse-Witt invariants of (Q', Q) are defined as follows.

- (1) $HW_1(Q', Q) := \deg_*(\text{inv}(Q', Q)) \in H^1(F, \mathbb{F}_2)$,
- (2) $HW_2(Q', Q) := \partial(\text{inv}(Q', Q)) \in \text{Br}_2(F)$,
- (3) $HW(Q', Q) := (HW_1(Q', Q), HW_2(Q', Q)) \in \text{Br}_2(F)_s$.

The following facts are helpful when one needs to manipulate Hasse-Witt invariants.

Lemma C.2. *For any three quadratic spaces (V, Q) , (V', Q') , (V'', Q'') of the same dimension there are equalities*

- (1) $HW(Q'', Q) = HW(Q'', Q') + HW(Q', Q)$,
- (2) $HW(Q', Q) = \text{Wall}(Q) - \text{Wall}(Q')$,

in the abelian group $\text{Br}_2(F)_s$.

C.2. Comparison of $\text{Wall}_1(Q)$, $\text{Wall}_2(Q)$ with the discriminant and classical Hasse-Witt invariant. In this paper we are primarily concerned with $\text{Wall}(Q)$ and $HW(Q', Q)$, which have the advantage of working in a uniform way for all fields, even those of characteristic 2. In Appendix G, however, we will need to recast a result of Fröhlich, who excluded characteristic 2 and used invariants that make sense only in that context. This subsection reviews the dictionary between the two systems.

In this subsection we assume that the characteristic of F is not 2. For $a_1, \dots, a_n \in F^\times$ we consider the quadratic form

$$Q(x_1, \dots, x_n) = a_1x_1^2 + a_2x_2^2 + \dots + a_nx_n^2.$$

We then have two classical invariants of Q , the discriminant

$$\text{disc}(Q) = \prod_{j=1}^n a_j \in F^\times / (F^\times)^2 = H^1(F, \mathbb{F}_2),$$

and the Hasse-Witt invariant

$$hw(Q) = \sum_{1 \leq i < j \leq n} a_i \cup a_j \in H^2(F, \mathbb{F}_2) = \text{Br}_2(F).$$

The invariants $\text{disc}(Q)$ and $hw(Q)$ are used in [Frö85]. In the case of quadratic forms of even rank, the Wall classes $\text{Wall}_1(Q)$, $\text{Wall}_2(Q)$ are different from, but carry the same information as, the invariants $\text{disc}(Q)$ and $hw(Q)$. In order to formulate a precise statement, we define an element $SW(Q) \in \text{Br}_2(F)_s$ by putting $SW(Q) = (\text{disc}(Q), hw(Q))$. The notation SW is meant to indicate that this is a truncated version of Delzant's [Del62] total Stiefel-Whitney class of Q , which lies in the multiplicative group

$$1 + H^1(F, \mathbb{F}_2) + H^2(F, \mathbb{F}_2) + H^3(F, \mathbb{F}_2) + \dots$$

We also define a specific element z in $\text{Br}_2(F)_s$ by $z := (z_1, z_2)$ with $z_1 = -1_F \in F^\times / (F^\times)^2 = H^1(F, \mathbb{F}_2)$ and $z_2 = 0 \in \text{Br}_2(F)$. Observe that z is the class $SW(Q_{F \times F})$ of the hyperbolic plane $(F \times F, Q_{F \times F})$.

Lemma C.3 (Wall). *Let (V, Q) be a quadratic space of dimension $2n$. Then there is an equality*

$$SW(Q) + \text{Wall}(Q) = nz$$

in the abelian group $\text{Br}_2(F)_s$.

Corollary C.4 (Wall). *Let (V, Q) , (V', Q') be two quadratic spaces of the same even dimension. Then there is an equality*

$$HW(Q', Q) = SW(Q') - SW(Q)$$

in the abelian group $\text{Br}_2(F)_s$.

Proof. Corollary C.4 follows from Lemma C.3 together with Lemma C.2(2). \square

APPENDIX D. REVIEW OF WEIL INDICES

In this appendix we assume our ground field F is local and fix a non-trivial additive character ψ on F . We again make the tacit assumption that all the quadratic spaces we consider are nondegenerate.

D.1. Weil indices for even dimensional quadratic spaces. For any quadratic space (V, Q) its Weil index $\gamma(Q, \psi)$ is defined. It is a complex eighth root of unity. When V is even dimensional the Weil index is actually a fourth root of unity.

Weil [Wei64] proved the following properties of the Weil index.

Proposition D.1 (Weil).

- (1) $\gamma(Q_1 \oplus Q_2, \psi) = \gamma(Q_1, \psi)\gamma(Q_2, \psi)$.
- (2) $\gamma(Q, \psi) = 1$ when Q is a hyperbolic plane.
- (3) $Q \mapsto \gamma(Q, \psi)$ induces a unitary character γ_ψ on the abelian group $W_0(F)$.
- (4) For any quaternion algebra D the Weil index of Q_D is given by

$$\gamma(Q_D, \psi) = \begin{cases} -1 & \text{if } D \text{ is not split,} \\ 1 & \text{if } D \text{ is split.} \end{cases}$$

Jacquet-Langlands [JL70, Lemma 1.2] proved the following result. As in Tate's article [Tat79] we write ϵ_L for Langlands's version of local epsilon factors. Thus $\epsilon_L(\rho, \psi)$ is defined for any continuous complex representation ρ of Γ (or, more generally, of the Weil group of F).

Proposition D.2 (Jacquet-Langlands). *Let E be a quadratic étale algebra, and let χ_E be the associated sign character on Γ . Then*

$$\gamma(Q_E, \psi) = \epsilon_L(\chi_E, \psi).$$

Proposition D.1 (4) and Proposition D.2 can be combined into a single statement. For this we define objects χ_Q and ζ_Q associated to any even dimensional quadratic space (V, Q) . First, χ_Q is by definition the sign character on Γ obtained as $\text{Wall}_1(Q)$. Second, ζ_Q is given by

$$\zeta_Q = \begin{cases} -1 & \text{if } \text{Wall}_2(Q) \text{ is nontrivial,} \\ 1 & \text{if } \text{Wall}_2(Q) \text{ is trivial.} \end{cases}$$

We should recall that $\text{Wall}_2(Q)$ lies in $\text{Br}_2(F)$, which for a local field is

- (1) cyclic of order 2 except when F is isomorphic to \mathbb{C} ,
- (2) trivial when F is isomorphic to \mathbb{C} .

Proposition D.3. *Let (V, Q) be an even dimensional quadratic space. Then*

$$\gamma(Q, \psi) = \epsilon_L(\chi_Q, \psi) \zeta_Q.$$

Proof. Unless F is isomorphic to \mathbb{R} the Wall homomorphism $W_0(F) \rightarrow \text{Br}_2(F)_s$ is an isomorphism, and the proposition we are now proving follows directly from Proposition D.1 (4) and Proposition D.2. The proof

in the real case is the same, once one observes that the unitary character γ_ψ on $W_0(\mathbb{R})$ is trivial on the kernel of the Wall homomorphism. Indeed the kernel of the Wall homomorphism is the infinite cyclic group generated by $Q_D \oplus Q_D$, where D is the Hamiltonian quaternion algebra, and $\gamma_\psi(Q_D \oplus Q_D) = (-1)^2 = 1$. \square

APPENDIX E. ORTHOGONAL REPRESENTATIONS OF Γ

Let ρ be a (continuous) finite dimensional complex representation of Γ . Assume that ρ is orthogonal. Then the Stiefel-Whitney classes $w_n(\rho) \in H^n(F, \mathbb{F}_2)$ are defined (see Deligne [Del76]). In fact they are even defined for virtual orthogonal complex representations, i.e. formal differences $\rho = \rho_1 - \rho_2$ of two orthogonal complex representations ρ_1 and ρ_2 .

We write $\bar{w}_2(\rho)$ for the element in $\text{Br}_2(F)$ obtained as the image of $w_2(\rho)$ under the homomorphism $H^2(F, \mathbb{F}_2) \rightarrow H^2(F, \overline{F}^\times)$ induced by the homomorphism $\mathbb{F}_2 \rightarrow \overline{F}^\times$ that sends n to $(-1)^n$. Of course $\bar{w}_2(\rho)$ is always trivial when the characteristic of F is 2. Finally, we put $\overline{SW}(\rho) = (w_1(\rho), \bar{w}_2(\rho)) \in \text{Br}_2(F)_s$.

Following Deligne [Del76], we want to write a formula for the local epsilon factor $\epsilon_L(\rho, \psi)$ in terms of the first and second Stiefel-Whitney classes of ρ . To do so we define objects χ_ρ and ζ_ρ . First, χ_ρ is by definition the sign character on Γ obtained as $w_1(\rho)$. More concretely, χ_ρ is the sign character $\det(\rho)$. Second, ζ_ρ is given by

$$\zeta_\rho = \begin{cases} -1 & \text{if } \bar{w}_2(\rho) \text{ is nontrivial,} \\ 1 & \text{if } \bar{w}_2(\rho) \text{ is trivial.} \end{cases}$$

We then have the following reformulation of Deligne's result:

Proposition E.1 (Deligne). *Let ρ be a virtual orthogonal representation of Γ . Then*

$$\epsilon_L(\rho, \psi) = \epsilon_L(\chi_\rho, \psi) \zeta_\rho.$$

Proof. Deligne proves this statement when ρ has dimension 0 and trivial determinant. In that case, since $\epsilon_L(\chi, \psi) = 1$ when χ is the trivial character on Γ , the proposition simply says that

$$\epsilon_L(\rho, \psi) = \zeta_\rho.$$

The general case follows from Deligne's result for the virtual representation $\rho' = \rho - \det(\rho) - V$, where V is a direct sum of $\dim(\rho) - 1$ copies of the trivial representation of Γ . The point is that $w_2(\rho') = w_2(\rho)$. \square

APPENDIX F. RELATION BETWEEN WEIL INDICES AND LOCAL
EPSILON FACTORS

Comparing Propositions D.3 and E.1, we arrive at the following conclusion. We again make the tacit assumption that the quadratic spaces we consider are nondegenerate.

Proposition F.1. *Let (V, Q) be an even dimensional quadratic space, and let ρ be a virtual orthogonal complex representation of Γ . Assume further that $\text{Wall}(Q) = \overline{SW}(\rho)$. Then*

$$\gamma(Q, \psi) = \epsilon_L(\rho, \psi).$$

Corollary F.2. *Let (V, Q) , (\tilde{V}, \tilde{Q}) be two quadratic spaces of the same even dimension, and let ρ be a virtual orthogonal complex representation of Γ . If $HW(Q, \tilde{Q}) = \overline{SW}(\rho)$, then $\gamma(\tilde{Q}, \psi)\gamma(Q, \psi)^{-1} = \epsilon_L(\rho, \psi)$.*

Proof. Apply the proposition to the quadratic form $\tilde{Q} \oplus (-Q)$. \square

APPENDIX G. FRÖHLICH THEORY IN ARBITRARY CHARACTERISTIC

Let F be a field, \overline{F} a separable closure, and $\Gamma = \text{Gal}(\overline{F}/F)$. We are again interested in the abelian groups $\text{Br}_2(F) \subset \text{Br}_2(F)_s$. We continue to make the tacit assumption that the quadratic spaces we consider are nondegenerate.

G.1. The local symbol $\{\chi, b\}$. We need to review the local symbol $\{\chi, b\}$ (a standard reference is [Ser62, Chapter 14]) in greater detail. For $\chi \in H^1(\Gamma, \mathbb{F}_2)$ and $b \in F^\times$, we denote by $\{\chi, b\}$ the element in $\text{Br}_2(F)$ obtained as the cup-product $b \cup \delta\chi$, where δ denotes the connecting homomorphism $H^1(F, \mathbb{F}_2) \rightarrow H^2(F, \mathbb{Z})$ obtained from the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{F}_2 \rightarrow 0$$

of (trivial) Γ -modules. It is evident that $\{\chi, b\}$ is additive in both χ and b . It follows that $\{\chi, b^2\} = (2\chi, b) = (0, b) = 0$, so that (χ, b) can be viewed as a pairing

$$(G.1) \quad H^1(\Gamma, \mathbb{F}_2) \otimes_{\mathbb{Z}} F^\times / (F^\times)^2 \rightarrow \text{Br}_2(F).$$

When the characteristic of F is not 2, Kummer theory identifies $F^\times / (F^\times)^2$ with $H^1(\Gamma, \mathbb{F}_2)$, and we may view the pairing above as the Hilbert pairing

$$F^\times / (F^\times)^2 \otimes_{\mathbb{Z}} F^\times / (F^\times)^2 \rightarrow \text{Br}_2(F)$$

or as the cup-product pairing

$$H^1(\Gamma, \mathbb{F}_2) \otimes_{\mathbb{Z}} H^1(\Gamma, \mathbb{F}_2) \rightarrow H^2(\Gamma, \mathbb{F}_2) = \text{Br}_2(F).$$

All of this is explained in Serre's book.

We are especially interested in characteristic 2, so the relevant pairing is (G.1). In order to formulate Fröhlich's result in a way that is valid even in characteristic 2, we need to view the pairing in an equivalent form. Let U be an \mathbb{F}_2 -vector space, and view U as a Γ -module with trivial Γ -action. There are obvious (cup-product) homomorphisms

$$H^n(\Gamma, \mathbb{F}_2) \otimes_{\mathbb{Z}} U \rightarrow H^n(\Gamma, U).$$

We claim that these homomorphisms are isomorphisms. Indeed, this follows from the fact that taking cohomology for a profinite group preserves direct sums (because it preserves finite direct sums and filtered colimits). Taking U to be $F^\times / (F^\times)^2$, we find that

$$H^1(\Gamma, F^\times / (F^\times)^2) = H^1(\Gamma, \mathbb{F}_2) \otimes_{\mathbb{Z}} F^\times / (F^\times)^2,$$

so that we may view the pairing (G.1) as a homomorphism

$$(G.2) \quad \xi : H^1(\Gamma, F^\times / (F^\times)^2) \rightarrow \text{Br}_2(F).$$

When we do so, we have the equality

$$(G.3) \quad \{\chi, b\} = \xi(\chi \otimes b).$$

We are going to use ξ to extend Fröhlich's result to fields of characteristic 2. For such a field the squaring map is injective, so square roots are unique when they exist. The next result gives a more explicit description of ξ in characteristic 2.

Lemma G.1. *Assume that the field F has characteristic 2. Let η be a homomorphism $\Gamma \rightarrow F^\times / (F^\times)^2$ with open kernel. Choose a 1-cochain u_σ of Γ in F^\times such that the image of u_σ under $F^\times \twoheadrightarrow F^\times / (F^\times)^2$ is $\eta(\sigma)$. The homomorphism property of η implies that $u_\sigma u_\tau u_{\sigma\tau}^{-1}$ is a square in F^\times , and $\xi(\eta)$ is represented by the 2-cocycle*

$$z_{\sigma,\tau} = \sqrt{u_\sigma u_\tau u_{\sigma\tau}^{-1}} \in F^\times \subset \overline{F}^\times.$$

Proof. Without loss of generality we may assume that η is of the form $\chi \otimes a$ with $\chi \in H^1(\Gamma, \mathbb{F}_2)$ and $a \in F^\times$. Concretely this means that $\eta(\tau) = a^{\chi(\tau)} \in F^\times / (F^\times)^2$.

We need to compute $\xi(\eta)$. To obtain a cocycle version of the connecting map $H^1(\Gamma, \mathbb{F}_2) \rightarrow H^2(\Gamma, \mathbb{Z})$, one needs to choose a set-theoretic section of $\mathbb{Z} \twoheadrightarrow \mathbb{F}_2$; our choice will be the liftings $0, 1 \in \mathbb{Z}$ of $0, 1 \in \mathbb{F}_2$. With this choice, $\xi(\eta) \in \text{Br}_2(F)$ is represented by the 2-cocycle z' of Γ in $F^\times \subset \overline{F}^\times$ given by

$$z'_{\sigma,\tau} = \begin{cases} a & \text{if both } \chi(\sigma) \text{ and } \chi(\tau) \text{ are nontrivial,} \\ 1 & \text{otherwise.} \end{cases}$$

Since the cohomology class of $z_{\sigma,\tau}$ is obviously independent of the choice of lifting u_σ , we are free to take

$$u_\sigma = a^{\dot{\chi}(\sigma)} \in F^\times,$$

where $\dot{\chi}(\sigma)$ is the unique element of $\{0, 1\} \subset \mathbb{Z}$ lifting $\chi(\sigma) \in \mathbb{F}_2$. The 2-cocycle

$$z_{\sigma,\tau} = \sqrt{u_\sigma u_\tau u_{\sigma\tau}^{-1}}$$

then coincides with z' , and the lemma is proved. \square

G.2. Stiefel-Whitney classes for orthogonal representations in arbitrary characteristic. Let k be an algebraically closed field, and let G be a profinite group. When the characteristic of k is not 2, Fröhlich attaches first and second Stiefel-Whitney classes to orthogonal representations of G on quadratic spaces over k . In other words, given a quadratic space (V, Q) over k , and a homomorphism $\varphi : G \rightarrow O(Q)$ with open kernel, Fröhlich defines an element $SW(\varphi)$ in the abelian group $1 + H^1(G, \mathbb{F}_2) + H^2(G, \mathbb{F}_2)$. The i -component of $SW(\varphi)$ ($i = 1, 2$) will be denoted by $SW_i(\varphi)$. For $k = \mathbb{C}$ these classes are the traditional Stiefel-Whitney classes w_1, w_2 that Deligne uses in his paper on local epsilon factors for orthogonal representations.

Fröhlich defines the first Stiefel-Whitney class $SW_1(\varphi)$ to be the sign character $\det \circ \varphi$. At the moment we are assuming that the characteristic of k is not 2, so there is a short exact sequence

$$1 \rightarrow \mu_2(k) \rightarrow \widetilde{\text{Pin}}(Q) \rightarrow O(Q) \rightarrow 1$$

of groups with trivial G -action, giving rise to a connecting map

$$\partial : H^1(G, O(Q)) \rightarrow H^2(G, \mu_2(k)) = H^2(G, \mathbb{F}_2),$$

and Fröhlich defines $SW_2(\varphi)$ to be the image of φ under this connecting map. (Here φ is viewed as a 1-cocycle of G in $O(Q)$.)

Now we want to extend Fröhlich's definitions to fields k of characteristic 2. We put $SW_1(\varphi) := \deg \circ \varphi$, where \deg is the degree homomorphism from $O(Q)$ to \mathbb{F}_2 . When the characteristic is not 2, \deg is equal to \det , so our definition is compatible with Fröhlich's. We put $SW_2(\varphi) = 0$ when k has characteristic 2. There is no real content in this definition, but it will allow for a uniform statement when we extend Fröhlich's result to characteristic 2.

G.3. Review of the spinor norm homomorphism. We consider a quadratic space (V, Q) over F . We are interested in its orthogonal group $O = O(Q)$ and Clifford group $Cl = Cl(Q)$. There is a short exact sequence (see equation (B.1))

$$1 \rightarrow F^\times \rightarrow Cl(F) \xrightarrow{\rho} O(F) \rightarrow 1,$$

as well as a canonical character $N : Cl(F) \rightarrow F^\times$ (see equation (B.3)).

Recall that the spinor norm homomorphism

$$\delta : O(F) \rightarrow F^\times / (F^\times)^2$$

is defined as follows. Let $x \in O(F)$ and choose $\dot{x} \in Cl(F)$ such that $\rho(\dot{x}) = x$. Then take $\delta(x)$ to be the square-class of $N(\dot{x})$.

G.4. Fröhlich's result. We want to extend a result of Fröhlich [Frö85] to fields F of arbitrary characteristic. Now consider an orthogonal representation $\varphi : \Gamma \rightarrow O(F)$. Once again Γ is the absolute Galois group of F . There are two things we can do with φ . The first is to extend scalars from F to an algebraically closed field k containing F . In this way we obtain an orthogonal representation

$$\varphi_k : \Gamma \xrightarrow{\varphi} O(F) \hookrightarrow O(k)$$

of Γ over k . It is evident that the Stiefel-Whitney class $SW(\varphi_k)$ is independent of the choice of k , so we may as well abbreviate it to $SW(\varphi)$. We denote by $\overline{SW}(\varphi)$ the image of $SW(\varphi)$ under the natural homomorphism from $1 + H^1(\Gamma, \mathbb{F}_2) + H^2(\Gamma, \mathbb{F}_2)$ to $\text{Br}_2(F)_s$.

The second thing we can do with φ is to twist (V, Q) (see subsection G.6 for a review), obtaining a new quadratic space (V_φ, Q_φ) over F . The element $\text{inv}(Q_\varphi, Q) \in H^1(F, O)$ is obtained as the image of φ under the natural map

$$H^1(\Gamma, O(F)) \rightarrow H^1(F, O).$$

We can then consider the relative Hasse-Witt invariant $HW(Q_\varphi, Q) \in \text{Br}_2(F)_s$.

The following result of Fröhlich makes use of the spinor norm map $\delta : O(F) \rightarrow F^\times / (F^\times)^2$ and the homomorphism $\xi : H^1(\Gamma, F^\times / (F^\times)^2) \rightarrow \text{Br}_2(F)$ defined earlier.

Proposition G.2 (Fröhlich). *$\overline{SW}(\varphi)$ and $HW(Q_\varphi, Q)$ differ by the element $\xi(\delta \circ \varphi)$ in the subgroup $\text{Br}_2(F)$ of $\text{Br}_2(F)_s$.*

Proof. When the characteristic is not 2, an equivalent result is proved by Fröhlich. When his result is restated in terms of relative Hasse-Witt invariants, it becomes the slightly different looking statement given here, as one can check using Corollary C.4.

Now we deal with the case when the characteristic is 2. It is clear that $SW_1(\varphi) = HW_1(Q_\varphi, Q)$. So the point is really to show that the elements $\overline{SW}_2(\varphi), HW_2(Q_\varphi, Q) \in \text{Br}_2(F)$ differ by $\xi(\eta)$, with $\eta := \delta \circ \varphi$.

Now $SW_2(\varphi)$ and $\overline{SW}_2(\varphi)$ are 0, so we need to prove that

$$HW_2(Q_\varphi, Q) = \xi(\eta).$$

To do so we make use of the exact sequence

$$1 \rightarrow F^\times \rightarrow Cl(F) \xrightarrow{\rho} O(F) \rightarrow 1.$$

We choose a finite Galois extension K/F such that φ is inflated from a homomorphism $\text{Gal}(K/F) \rightarrow O(F)$. For $\sigma \in \text{Gal}(K/F)$ we choose $\dot{\varphi}(\sigma) \in Cl(F)$ such that $\rho(\dot{\varphi}(\sigma)) = \varphi(\sigma)$. Since these liftings are fixed by the Galois group, the Hasse-Witt invariant $HW_2(Q_\varphi, Q)$ is represented by the 2-cocycle

$$(G.4) \quad z_{\sigma, \tau} = \dot{\varphi}(\sigma) \dot{\varphi}(\tau) (\dot{\varphi}(\sigma\tau))^{-1}.$$

Next we need to calculate $\xi(\eta)$. Applying the character N to the equation (G.4), we find that

$$z_{\sigma, \tau}^2 = u_\sigma u_\tau u_{\sigma\tau}^{-1},$$

with $u_\sigma \in F^\times$ defined by $u_\sigma = N(\dot{\varphi}(\sigma))$. Now $\eta(\sigma)$ is equal to the square class of u_σ . So Lemma G.1 implies that $\xi(\eta)$ is represented by the 2-cocycle

$$\sqrt{u_\sigma u_\tau u_{\sigma\tau}^{-1}} = z_{\sigma, \tau}.$$

Thus $z_{\sigma, \tau}$ represents both $HW_2(Q_\varphi, Q)$ and $\xi(\eta)$, and the proof is complete. \square

G.5. Spinor norms of reflections. When applying Fröhlich's result, it is useful to understand spinor norms of reflections. For $v \in V$ with $Q(v) \neq 0$, the element v in the Clifford algebra $C(Q)$ is an odd element in the Clifford group. It projects to the reflection $r_v \in O(Q)$. So the spinor norm of r_v is $N(v) = v^2 = Q(v)$.

G.6. Review of twisting of quadratic forms. Let V be an F -vector space. Then $\overline{F} \otimes_F V$ carries a semilinear smooth Γ -action, given by $\sigma(\alpha \otimes v) = \sigma(\alpha) \otimes v$, having $V \subset \overline{F} \otimes_F V$ as fixed point set. Conversely, given a vector space U over \overline{F} , equipped with a semilinear smooth action of Γ , we obtain an F -vector space $V = U^\Gamma$ and a canonical isomorphism $\overline{F} \otimes_F V \rightarrow U$. Informally, then, giving an F -vector space is the same as giving an \overline{F} -vector space equipped with a semilinear smooth Galois action. This is Galois descent theory.

In particular, given an F -vector space V and a continuous 1-cocycle $\sigma \mapsto \theta_\sigma$ of Γ in $\text{Aut}(V)(\overline{F})$, we get a new semilinear action $\sigma_* := \theta_\sigma \sigma$, and so get a new F -vector space

$$V_* = \{v \in \overline{F} \otimes_F V : \theta_\sigma(\sigma(v)) = v \quad \forall \sigma \in \Gamma\}$$

having the same dimension as V . One says that V_* is a *twist* of V .

Now suppose that we have a quadratic form Q on V , and suppose further that our 1-cocycle θ_σ takes values in the subgroup $O(Q)(\overline{F})$ of

$\text{Aut}(V)(\overline{F})$. Because Q is defined over F , it is fixed by σ . Because θ_σ fixes Q , the action of σ_* also fixes Q . In particular, the function $Q : \overline{F} \otimes_F V \rightarrow \overline{F}$ is F -valued on V_* as well as V . In summary, twisting a quadratic space (V, Q) actually means that we twist the F -structure on V , while leaving the quadratic form unchanged. Nevertheless, we speak of “twisting the quadratic form Q by the 1-cocycle θ ” and denote the twisted form by Q_θ .

APPENDIX H. RELATIVE HASSE-WITT INVARIANTS FOR TWISTS BY 1-COCYCLES IN TORI

Let (V, Q) be a nondegenerate quadratic space over F with V even dimensional. Let T be a torus, and suppose that we are given an action of T on V by orthogonal transformations. In other words we are given an F -homomorphism from T to the orthogonal group of V . Finally, suppose that we are given a 1-cocycle t of Γ in T . We can then twist (V, Q) by t , obtaining a new quadratic form Q_t .

Now we are going to use t to define an element in $\text{Br}_2(F)$. We begin by defining an extension U of T by μ_2 . Such extensions are classified by Γ -invariant elements in the \mathbb{F}_2 -vector space $X^*(T)/2X^*(T)$. We now use the orthogonal action of T on V to define a canonical such element, which we will denote by λ_V .

Let us write the dimension of V as $n = 2m$. Because V is self-contragredient as representation of T , we have $\dim V_\lambda = \dim V_{-\lambda}$ for each $\lambda \in X^*(T)$. (Here V_λ is the weight space of λ in V ; it is defined over the separable closure.) So there exist $\lambda_1, \dots, \lambda_m \in X^*(T)$ such that the n characters of T occurring in V are $\lambda_1, \dots, \lambda_m, -\lambda_1, \dots, -\lambda_m$. The m -tuple $(\lambda_1, \dots, \lambda_m)$ is well-defined up to permutations and sign changes. Therefore the sum $\lambda_V := \lambda_1 + \dots + \lambda_m$ yields a well-defined element of $X^*(T)/2X^*(T)$. Our assumption that the torus action is defined over F implies that λ_V is a Γ -invariant element in $X^*(T)/2X^*(T)$.

Now, as we have already said, to λ_V is associated an extension

$$1 \rightarrow \mu_2 \rightarrow U \rightarrow T \rightarrow 1.$$

This sequence of F -groups is exact in the fppf topology, so from it we obtain a connecting homomorphism

$$\partial_U : H_{\text{fppf}}^1(F, T) \rightarrow H_{\text{fppf}}^2(F, \mu_2).$$

The source coincides with the Galois cohomology group $H^1(F, T)$, and the target coincides with $\text{Br}_2(F)$, so we may also view ∂_U as a homomorphism

$$(H.1) \quad \partial_U : H^1(F, T) \rightarrow \text{Br}_2(F).$$

Lemma H.1. *$HW(Q_t, Q)$ is equal to the image of t under the map (H.1). Here, as usual, we are viewing $\mathrm{Br}_2(F)$ as a subgroup of $\mathrm{Br}_2(F)_s$.*

Proof. We must prove that

- (1) $HW_1(Q_t, Q) = 0$,
- (2) $HW_2(Q_t, Q) = \partial_U(t)$.

Without loss of generality we may assume that T is a maximal torus in $O(Q)$. The first item follows from the fact that T lies in the kernel $SO(Q)$ of the degree homomorphism $O(Q) \rightarrow \mathbb{F}_2$. The second item follows from the well-known fact that the element of $X^*(T)/2X^*(T)$ associated to

$$1 \rightarrow \mu_2 \rightarrow \mathrm{Spin}(Q) \rightarrow O(Q) \rightarrow 1$$

is λ_V . □

Corollary H.2. *Now assume that F is local, so that Weil indices are defined. Then*

$$\gamma(Q_t, \psi)\gamma(Q, \psi)^{-1} = \partial_U(t) \in \mathrm{Br}_2(F) = \{\pm 1\}.$$

Proof. This follows from Lemma H.1 together with Lemmas D.3 and C.2(2). □

APPENDIX I. STANDARD QUADRATIC FORM Q_2 ON $\mathrm{Lie}(\mathbb{G}_{\mathrm{sc}})$ OVER \mathbb{Z}

I.1. Notation. In this appendix we consider a split group \mathbb{G}_{sc} over \mathbb{Z} that is semisimple and simply connected. In it we choose a split maximal torus \mathbb{T}_{sc} , and a Borel subgroup \mathbb{B}_{sc} containing \mathbb{T}_{sc} . For each simple root α of \mathbb{T}_{sc} we choose an isomorphism η_α from the additive group \mathbb{G}_a to the root subgroup \mathbb{G}_α . Thus $(\mathbb{G}_{\mathrm{sc}}, \mathbb{B}_{\mathrm{sc}}, \mathbb{T}_{\mathrm{sc}}, \{\eta_\alpha\})$ is a pinned group over \mathbb{Z} . Moreover, we assume that the root system R of \mathbb{T}_{sc} in \mathbb{G}_{sc} is irreducible. We write $Z(\mathbb{G}_{\mathrm{sc}})$ for the center of \mathbb{G}_{sc} , and \mathbb{G}_{ad} for the adjoint group $\mathbb{G}_{\mathrm{sc}}/Z(\mathbb{G}_{\mathrm{sc}})$.

We write W for the Weyl group of \mathbb{T}_{sc} in \mathbb{G}_{sc} , and put

$$\Omega_0 := \mathrm{Aut}(\mathbb{G}_{\mathrm{sc}}, \mathbb{B}_{\mathrm{sc}}, \mathbb{T}_{\mathrm{sc}}, \{\eta_\alpha\}).$$

It is well-known that $\mathrm{Aut}(\mathbb{G}_{\mathrm{sc}}) = \mathbb{G}_{\mathrm{ad}} \rtimes \Omega_0$. The group $\Omega := W \rtimes \Omega_0$ acts on \mathbb{T}_{sc} .

I.2. Standard quadratic form Q_1 on $\mathrm{Lie}(\mathbb{T}_{\mathrm{sc}})$. We put

$$\mathfrak{a} := X_*(\mathbb{T}_{\mathrm{sc}})_{\mathbb{Q}} = X_*(\mathbb{T}_{\mathrm{ad}})_{\mathbb{Q}}.$$

So we are now viewing

$$X_*(\mathbb{T}_{\mathrm{sc}}) \subset X_*(\mathbb{T}_{\mathrm{ad}})$$

as an inclusion of lattices in a common rational vector space \mathfrak{a} .

Because the root system R is irreducible, the space of W -invariant quadratic forms on \mathfrak{a} is one dimensional. As basis element we choose the unique W -invariant quadratic form Q_1 whose values on coroots lie in the set $\{1, \ell\}$, where

$$\ell = \begin{cases} 1 & \text{if } R \text{ is of type } A, D, E, \\ 2 & \text{if } R \text{ is of type } B_n, C_n, F_4 \text{ with } n \geq 2, \\ 3 & \text{if } R \text{ is of type } G_2. \end{cases}$$

So, when all coroots have the same length, Q_1 takes the value 1 on them. Otherwise Q_1 takes the value 1 on short coroots, and takes the value ℓ on long coroots. The action of Ω preserves Q_1 . We put $\ell(\alpha^\vee) := Q_1(\alpha^\vee) \in \{1, \ell\}$.

Observe that ℓ is the same for R^\vee as it is for R . The bijection $\alpha \mapsto \alpha^\vee$ from R to R^\vee sends long roots to short coroots, and vice versa, so there is an equality

$$(I.1) \quad \ell(\alpha^\vee)\ell(\alpha) = \ell.$$

Now polarize Q_1 to obtain an Ω -invariant symmetric bilinear form B_1 on \mathfrak{a} . Thus B_1 and Q_1 are related by the identities $B_1(x, y) = Q_1(x + y) - Q_1(x) - Q_1(y)$ and $Q_1(x) = \frac{1}{2}B_1(x, x)$. We may also view B_1 as an isomorphism $B_1 : \mathfrak{a} \rightarrow \mathfrak{a}^*$. It will always be clear from context which interpretation of B_1 is meant; the two interpretations are related by the identity

$$B_1(x, y) = \langle B_1(x), y \rangle.$$

(We always use $\langle \cdot, \cdot \rangle$ to denote the canonical pairing between a module and its linear dual.)

From now on we use B_1 to identify \mathfrak{a}^* with \mathfrak{a} . This allows us to view roots as elements in \mathfrak{a} .

Lemma I.1. *The following statements hold.*

- (1) *There is an equality $\alpha^\vee = \ell(\alpha^\vee)\alpha \in \mathfrak{a}$.*
- (2) *There is an equality $\ell Q_1(\alpha) = \ell(\alpha)$.*

Lemma I.2. *The integrality properties of Q_1 and B_1 are as follows.*

- (1) *Q_1 takes integral values on the lattice $X_*(T_{\text{sc}})$ in \mathfrak{a} .*
- (2) *$B_1(x, y) \in \mathbb{Z}$ when $x \in X_*(T_{\text{sc}})$, $y \in X_*(T_{\text{ad}})$.*
- (3) *ℓQ_1 takes integral values on the lattice $X^*(T_{\text{ad}})$ in \mathfrak{a} .*
- (4) *$\ell B_1(x, y) \in \mathbb{Z}$ when $x \in X^*(T_{\text{sc}})$, $y \in X^*(T_{\text{ad}})$.*

In the third and fourth parts, one must bear in mind that we have identified \mathfrak{a}^* with \mathfrak{a} , so that $X^*(T_{\text{sc}})$, $X^*(T_{\text{ad}})$ may be viewed as lattices in \mathfrak{a} .

I.3. The standard quadratic form Q_2 on $\mathrm{Lie}(\mathbb{G}_{\mathrm{sc}})$. The space of \mathbb{G}_{ad} -invariant quadratic forms on $\mathrm{Lie}(\mathbb{G}_{\mathrm{sc}})$ is an abelian group that is free of rank 1. It has a unique generator Q_2 which restricts to Q_1 on $\mathrm{Lie}(\mathbb{T}_{\mathrm{sc}}) = X_*(\mathbb{T}_{\mathrm{sc}})$. Moreover Q_2 is invariant under all automorphisms of \mathbb{G}_{sc} .

This is well-known, but we need to recall how it works. Start by extending scalars to \mathbb{Q} . The space of invariant quadratic forms on $\mathrm{Lie}(\mathbb{G}_{\mathrm{sc}})_{\mathbb{Q}}$ is 1-dimensional, and any nonzero invariant form (e.g. the Killing form) is nondegenerate over \mathbb{Q} . Also, any invariant form is actually invariant under all automorphisms of \mathbb{G}_{sc} , since this is obviously true for the Killing form.

Now we decompose the Lie algebra of \mathbb{G}_{sc} as

$$(I.2) \quad \mathrm{Lie}(\mathbb{G}_{\mathrm{sc}}) = \mathrm{Lie}(\mathbb{T}_{\mathrm{sc}}) \oplus \mathbb{V},$$

where

$$(I.3) \quad \mathbb{V} = \bigoplus_{\alpha > 0} (\mathrm{Lie}(\mathbb{G}_{\alpha}) \oplus \mathrm{Lie}(\mathbb{G}_{-\alpha})).$$

Both (I.2) and (I.3) are orthogonal decompositions with respect to any invariant quadratic form.

The restriction to $\mathrm{Lie}(\mathbb{T}_{\mathrm{sc}}) = X_*(\mathbb{T}_{\mathrm{sc}})$ of any nonzero invariant form is W -invariant, hence is a nonzero rational multiple of Q_1 . So it is clear that there exists a unique invariant quadratic form Q_2 on $\mathrm{Lie}(\mathbb{G}_{\mathrm{sc}})_{\mathbb{Q}}$ whose restriction to $\mathrm{Lie}(\mathbb{T}_{\mathrm{sc}}) = X_*(\mathbb{T}_{\mathrm{sc}})$ is Q_1 .

Next we are going to remind the reader why Q_2 takes integral values on the lattice $\mathrm{Lie}(\mathbb{G}_{\mathrm{sc}})$ in $\mathrm{Lie}(\mathbb{G}_{\mathrm{sc}})_{\mathbb{Q}}$. The restriction of Q_2 to $\mathrm{Lie}(\mathbb{T}_{\mathrm{sc}})$ is Q_1 , which takes integral values, so we just need to prove the integrality of the restriction of Q_2 to each subspace $\mathrm{Lie}(\mathbb{G}_{\alpha}) \oplus \mathrm{Lie}(\mathbb{G}_{-\alpha})$.

So we need to understand what happens for SL_2 , with diagonal maximal torus. Then Q_2 is the quadratic form whose value on the matrix

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

is the negative of its determinant, namely $a^2 + bc$. Observe that the restriction of Q_2 to the direct sum of the two root spaces is the quadratic form bc . We can put this answer in more intrinsic terms by observing that for $v = (x, y) \in \mathrm{Lie}(\mathbb{G}_{\alpha}) \oplus \mathrm{Lie}(\mathbb{G}_{-\alpha})$ the value of Q_2 on (x, y) is equal to c , where c is the unique integer such that the Lie bracket $[x, y]$ is equal to cH_{α} . (We write H_{α} for the coroot α^{\vee} when we are viewing it in $\mathrm{Lie}(\mathbb{T}_{\mathrm{sc}})$ rather than $X_*(\mathbb{T}_{\mathrm{sc}})$.) Moreover $Q_2(H_{\alpha}) = 1$ for both roots of SL_2 .

Returning to the general case, there is still an obvious \mathbb{T}_{sc} -invariant quadratic form $Q_{\pm\alpha}$ on $\text{Lie}(\mathbb{G}_\alpha) \oplus \text{Lie}(\mathbb{G}_{-\alpha})$, namely the one whose value on (x, y) is equal to the unique integer c such that the Lie bracket $[x, y]$ is equal to cH_α . The restriction of Q_2 to $\text{Lie}(\mathbb{G}_\alpha) \oplus \text{Lie}(\mathbb{G}_{-\alpha})$ is necessarily some scalar times $Q_{\pm\alpha}$. Our computation for SL_2 shows that the scalar must be the value of Q_2 on H_α , namely $\ell(\alpha^\vee)$. In any case Q_2 does take integral values on $\text{Lie}(\mathbb{G}_{\text{sc}})$.

I.4. Definition of the nondegenerate quadratic form $Q_{\mathbb{V}}$ on \mathbb{V} . In the last subsection, imitating what we did in subsection 1.3.3, we defined nondegenerate quadratic forms $Q_{\pm\alpha}$ for each unordered pair $\{\pm\alpha\}$. We now define a nondegenerate quadratic form $Q_{\mathbb{V}}$ on \mathbb{V} as the direct sum of the various $Q_{\pm\alpha}$. In the next subsection we compare $Q_{\mathbb{V}}$ with the restriction of Q_2 to \mathbb{V} .

I.5. The restriction of Q_2 to \mathbb{V} . Our review of how Q_2 is constructed gives us a good understanding of the restriction of Q_2 to \mathbb{V} . This restriction is equal to $Q_{\mathbb{V}}$ if and only if the root system is simply laced. When the root system is not simply laced, \mathbb{V} is the orthogonal direct sum of \mathbb{V}' and \mathbb{V}'' , where \mathbb{V}' is the direct sum of the root spaces for the various long roots, and \mathbb{V}'' is the direct sum of the root spaces for the various short roots. We denote by $Q_{\mathbb{V}'}$ (resp., $Q_{\mathbb{V}''}$) the restriction of $Q_{\mathbb{V}}$ to \mathbb{V}' (resp., \mathbb{V}''). The computations we made while discussing Q_2 show that

- the restriction of Q_2 to \mathbb{V}' is equal to $Q_{\mathbb{V}'}$,
- the restriction of Q_2 to \mathbb{V}'' is equal to $\ell Q_{\mathbb{V}''}$.

(Long roots correspond to short coroots, and on these Q_2 takes the value 1.)

In the simply laced case it is convenient to make the convention that $\mathbb{V}' = \mathbb{V}$ and $\mathbb{V}'' = 0$. With this convention we have $\mathbb{V} = \mathbb{V}' \oplus \mathbb{V}''$ in all cases.

APPENDIX J. SOME LEMMAS ABOUT LATTICES IN RATIONAL QUADRATIC SPACES

The next lemmas are quite general in nature. In the body of the paper we apply them to the lattices $\text{Lie}(\mathbb{G})$, $\text{Lie}(\mathbb{T})$, \mathbb{V} , \mathbb{V}' and \mathbb{V}'' .

J.1. Notation. In this appendix we consider a triple (Λ, Q, ℓ) that consists of

- a \mathbb{Z} -lattice Λ ,
- a nondegenerate quadratic form $Q : \Lambda_{\mathbb{Q}} \rightarrow \mathbb{Q}$,
- a positive integer ℓ ,

subject to the requirement that

- Q takes integral values on Λ , and
- ℓQ takes integral values on Λ^\perp .

As always, Λ^\perp is the perpendicular lattice with respect to the symmetric bilinear form B obtained by polarizing Q , i.e.

$$\Lambda^\perp = \{x \in \Lambda_{\mathbb{Q}} : B(x, y) \in \mathbb{Z} \quad \forall y \in \Lambda\}.$$

Once again we remind the reader that $B(x, y) = Q(x+y) - Q(x) - Q(y)$.

J.2. The induced quadratic forms on $\Lambda/\ell\Lambda^\perp$ and Λ^\perp/Λ .

Lemma J.1. *The following statements hold.*

- (1) $\ell\Lambda^\perp \subset \Lambda \subset \Lambda^\perp \subset \ell^{-1}\Lambda$.
- (2) Let $\pi' : \Lambda \twoheadrightarrow \Lambda/\ell\Lambda^\perp$ denote the canonical surjection. Then there exists a unique quadratic form $Q' : \Lambda/\ell\Lambda^\perp \rightarrow \mathbb{Z}/\ell\mathbb{Z}$ on the $\mathbb{Z}/\ell\mathbb{Z}$ -module $\Lambda/\ell\Lambda^\perp$ such that, for all $x \in \Lambda$, the value of Q' on $\pi'(x)$ is equal to the reduction modulo ℓ of $Q(x)$.
- (3) Let $\pi'' : \Lambda^\perp \twoheadrightarrow \Lambda^\perp/\Lambda$ denote the canonical surjection. Then there exists a unique quadratic form $Q'' : \Lambda^\perp/\Lambda \rightarrow \mathbb{Z}/\ell\mathbb{Z}$ on the $\mathbb{Z}/\ell\mathbb{Z}$ -module Λ^\perp/Λ such that, for all $x \in \Lambda^\perp$, the value of Q'' on $\pi''(x)$ is equal to the reduction modulo ℓ of $\ell Q(x)$.
- (4) The quadratic form Q on Λ becomes nondegenerate over $\mathbb{Z}[1/\ell]$.
- (5) If ℓ is prime, then the quadratic forms Q' and Q'' are nondegenerate over the field $\mathbb{Z}/\ell\mathbb{Z}$.

Proof. (1) Our assumption that Q takes integral values on Λ implies that B takes integral values on $\Lambda \times \Lambda$, which is to say that $\Lambda \subset \Lambda^\perp$. Similarly, our assumption that ℓQ takes integral values on Λ implies that $\Lambda^\perp \subset \ell^{-1}\Lambda$ (equivalently, that $\ell\Lambda^\perp \subset \Lambda$).

(2) We must check that $Q(x + \ell y)$ is congruent to $Q(x)$ modulo ℓ for all $x \in \Lambda$, $y \in \Lambda^\perp$. This is indeed so, because $Q(x + \ell y) - Q(x) = Q(\ell y) + B(x, \ell y)$, and

$$\begin{aligned} Q(\ell y) &= \ell(\ell Q(y)) \in \ell\mathbb{Z}, \\ B(x, \ell y) &= \ell B(x, y) \in \ell\mathbb{Z}. \end{aligned}$$

(3) follows from (2), applied to the quadratic form ℓQ and the lattice Λ^\perp . (The perpendicular of Λ^\perp with respect to ℓQ is $\ell^{-1}\Lambda$.)

(4) follows from the fact that, after tensoring with $\mathbb{Z}[1/\ell]$, the inclusions in (1) become equalities.

(5) We begin by proving that Q' is nondegenerate. The bilinear form B takes values in $\ell\mathbb{Z}$ when one of its arguments lies in Λ and the other in $\ell\Lambda^\perp$. Therefore B induces a symmetric bilinear form

$$B' : \Lambda/\ell\Lambda^\perp \times \Lambda/\ell\Lambda^\perp \rightarrow \mathbb{Z}/\ell\mathbb{Z},$$

and it is evident that B' coincides with the bilinear form obtained by polarizing Q' . We must show that B' is nondegenerate. Consider an element $\bar{x} \in \Lambda/\ell\Lambda^\perp$ that pairs trivially with all elements in $\Lambda/\ell\Lambda^\perp$, and represent \bar{x} by an element $x \in \Lambda$. Then $B(x, y)$ lies in $\ell\mathbb{Z}$ for all $y \in \Lambda$, so that $x \in \ell\Lambda^\perp$ and hence $\bar{x} = 0$.

The nondegeneracy of Q'' follows from the method used to derive (3) from (2). \square

J.3. The homomorphism $\text{Aut}(\Lambda, Q) \rightarrow O(Q') \times O(Q'')$. The next lemma concerns $\text{Aut}(\Lambda, Q)$, the group of linear automorphisms of $\Lambda_\mathbb{Q}$ that preserve both Q and Λ . In the lemma we assume that ℓ is prime. The previous lemma then provides nondegenerate quadratic forms

- Q' on the \mathbb{F}_ℓ -vector space $\Lambda/\ell\Lambda^\perp$, and
- Q'' on the \mathbb{F}_ℓ -vector space Λ^\perp/Λ .

Now suppose that we are given $g \in \text{Aut}(\Lambda, Q)$. Then g preserves Λ and Λ^\perp , so it induces a linear transformation g' on $\Lambda/\ell\Lambda^\perp$, as well as a linear transformation g'' on Λ^\perp/Λ . It is easy to see that both g' and g'' are orthogonal transformations, and of course $g \mapsto (g', g'')$ is a homomorphism from $\text{Aut}(\Lambda, Q)$ to $O(Q') \times O(Q'')$. In the next lemma we compute g', g'' for reflections $g = r_v$ in certain vectors v .

Lemma J.2. *Let $v \in \Lambda$ and suppose that $Q(v)$ is a unit. Consider the reflection $g := r_v$. Then*

- $g \in \text{Aut}(\Lambda, Q)$,
- $g' = r_{v'}$, where v' denotes the image of v under $\Lambda \twoheadrightarrow \Lambda/\ell\Lambda^\perp$,
- g'' is the identity element.

Similarly, now let $v \in \Lambda^\perp$ and suppose that $(\ell Q)(v)$ is a unit. Consider the reflection $g := r_v$. Then

- $g \in \text{Aut}(\Lambda, Q)$,
- g' is the identity element,
- $g'' = r_{v''}$, where v'' denotes the image of v under $\Lambda^\perp \twoheadrightarrow \Lambda^\perp/\Lambda$.

Proof. The second half of the lemma follows easily from the first. One just has to apply the first part with (Λ, Q) replaced by $(\Lambda^\perp, \ell Q)$. It remains to prove the first half of the lemma, so we consider $v \in \Lambda$ such that $Q(v)$ is a unit. Then r_v is given by the formula

$$r_v(x) = x - \frac{B(v, x)}{Q(v)}v.$$

Since $Q(v)$ is a unit, it is clear that r_v lies in $\text{Aut}(\Lambda, Q)$ and that $(r_v)' = r_{v'}$. It remains only to check that $(r_v)''$ is the identity. In other words, we must show that $r_v(x) - x$ lies in Λ whenever $x \in \Lambda^\perp$, and this is clear because $Q(v)$ is a unit, $B(v, x)$ is integral, and $v \in \Lambda$. \square

APPENDIX K. COMPARISON OF STIEFEL-WHITNEY CLASSES
BEFORE AND AFTER REDUCTION MODULO p

For the purposes of this paper we just need to understand how Stiefel-Whitney classes behave under reduction mod 2 and 3, but in this appendix we might as well treat a general prime number p . Consider a nondegenerate quadratic space (Λ, Q) over \mathbb{Z}_p . (When $p = 2$, the rank of the free module Λ is necessarily even.) Then $O(Q)$, which we abbreviate to O , is a smooth group scheme over \mathbb{Z}_p . We choose algebraic closures $\overline{\mathbb{Q}}_p, \overline{\mathbb{F}}_p$ of $\mathbb{Q}_p, \mathbb{F}_p$ respectively.

Let Γ be a profinite group, and suppose that we are given a homomorphism $\rho : \Gamma \rightarrow O(\mathbb{Z}_p)$ with open kernel. From it we obtain orthogonal representations $\rho_{\mathbb{Q}_p} : \Gamma \rightarrow O(\overline{\mathbb{Q}}_p)$ and $\rho_{\mathbb{F}_p} : \Gamma \rightarrow O(\overline{\mathbb{F}}_p)$.

Lemma K.1. *When $p \neq 2$, there is an equality*

$$SW(\rho_{\mathbb{Q}_p}) = SW(\rho_{\mathbb{F}_p})$$

of elements in $1 + H^1(\Gamma, \mathbb{F}_2) + H^2(\Gamma, \mathbb{F}_2)$. When $p = 2$ there is an equality $SW_1(\rho_{\mathbb{Q}_p}) = SW_1(\rho_{\mathbb{F}_p}) \in H^1(\Gamma, \mathbb{F}_2)$.

Proof. We begin with the first Stiefel-Whitney class, which is obtained by applying the sign character $\deg : O \rightarrow \mathbb{F}_2$ to the given orthogonal representation. Now \deg is a homomorphism of group schemes over \mathbb{Z}_p , so the diagram

$$\begin{array}{ccccc} O(\overline{\mathbb{F}}_p) & \longleftarrow & O(\mathbb{Z}_p) & \longrightarrow & O(\overline{\mathbb{Q}}_p) \\ \deg \downarrow & & \deg \downarrow & & \deg \downarrow \\ \mathbb{F}_2 & \xlongequal{\quad} & \mathbb{F}_2 & \xlongequal{\quad} & \mathbb{F}_2 \end{array}$$

commutes, which proves that $SW_1(\rho_{\mathbb{Q}_p}) = SW_1(\rho_{\mathbb{F}_p})$.

Now assuming that $p \neq 2$, we need to show that the second Stiefel-Whitney classes of $\rho_{\mathbb{Q}_p}$ and $\rho_{\mathbb{F}_p}$ coincide. For this we must compare the double covers $\widetilde{\text{Pin}}(k) \rightarrow O(k)$ for $k = \overline{\mathbb{Q}}_p$ and $k = \overline{\mathbb{F}}_p$. This is done using the commutative diagram

$$\begin{array}{ccccc} \widetilde{\text{Pin}}(\overline{\mathbb{F}}_p) & \longleftarrow & \widetilde{\text{Pin}}(\overline{\mathbb{Z}}_p) & \longrightarrow & \widetilde{\text{Pin}}(\overline{\mathbb{Q}}_p) \\ \downarrow & & \downarrow & & \downarrow \\ O(\overline{\mathbb{F}}_p) & \longleftarrow & O(\overline{\mathbb{Z}}_p) & \longrightarrow & O(\overline{\mathbb{Q}}_p), \end{array}$$

where $\overline{\mathbb{Z}}_p$ is the integral closure of \mathbb{Z}_p in $\overline{\mathbb{Q}}_p$. The middle vertical arrow is surjective because $\overline{\mathbb{Z}}_p$ is a strictly henselian local ring and $\widetilde{\text{Pin}} \rightarrow O$ is a surjective smooth morphism (by our assumption that $p \neq 2$). So

the middle arrow is a double cover whose kernel $\mu_2(\overline{\mathbb{Z}}_p)$ maps isomorphically to the kernels $\mu_2(\overline{\mathbb{F}}_p)$, $\mu_2(\overline{\mathbb{Q}}_p)$ of the other two double covers in the diagram. \square

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